NASA TECHNICAL TRANSLATION NASA TT F-17159

# NUMERICAL METHODS FOR THE STATISTICAL ANALYSIS OF RANDOM PROCESSES BY MEANS OF COMPUTERS

G. F. Telenin and F. V. Podsypanin

(NASA-TT-F-17159) NUMERICAL METHODS FOR THE STATISTICAL ANALYSIS OF RANDOM PROCESSES BY MEANS OF COMPUTERS (National Aeronautics and Space Administration) 63 p HC A04/MF A01 CSCL 09B G3/61

N78-18795

Unclas 05468

Translation of "Chislennye Metody Statisticheskogo Analiza Sluchainykh Protsessov s Pomoshch'yu ETsVM," Scientific Transactions, No. 6, Institute of Mechanics, Moscow State University, Moscow, 1970, pp 1-48

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 FEBRUARY 1978

STANDARD TITLE PAGE

| 1. Report No. | 2. Government Accession No. | 3. Recipient's Catalog No. |
|---|---|---|
| NASA TT F-17159 | NTIS TT 77-58121 | |

| 4. Title and Subtitle | 5. Report Date |
|---|---|
| NUMERICAL METHODS FOR THE STATISTICAL ANALYSIS OF RANDOM PROCESSES BY MEANS OF COMPUTERS | FEBRUARY 1978 |
| | 6. Performing Organization Code |

| 7. Author(s) | 8. Performing Organization Report No. |
|---|---|
| G. F. Telenin and F. V. Podsypanin | 10. Work Unit No. |
| | 11. Contract or Grant No. NSF C-724 |

| 9. Performing Organization Name and Address | 13. Type of Report and Period Covered |
|---|---|
| Franklin Book Programs Inc, Cairo, Egypt | Translation |
| 12. Sponsoring Agency Name and Address | 14. Sponsoring Agency Code |
| NATIONAL AERONAUTIC & SPACE ADMINISTRATION and NATIONAL SCIE NCE FOUNDATION, WASHINGTON, D.C. | |

15. Supplementary Notes

Translation of "Chislennye Metody Statisticheskogo Analiza Sluchaynykh Protsessov s Pomoshch'yu ETsVM," Scientific Translations, No. 6, Institute of Mechanics, Moscow State University, Moscow, 1970, pp 1-48

16. Abstract

| 17. Key Words (Selected by Author(s)) | 18. Distribution Statement |
|---|---|
| | Unlimited |

| 19. Security Classif. (of this report) | 20. Security Classif. (of this page) | 21. No. of Pages | 22. Price |
|---|---|---|---|
| | Unclassified | | |

TT 77-58121
NASA TT F-17159

Translated from Russian

LOMONOSOV'S MOSCOW STATE UNIVERSITY,
INSTITUTE OF MECHANICS

(Institut Mekhaniki)

SCIENTIFIC TRANSACTIONS
No. 6

(Nauchnye Trudy)
No. 6

# NUMERICAL METHODS FOR THE STATISTICAL ANALYSIS OF RANDOM PROCESSES BY MEANS OF COMPUTERS

(Chislennye Metody Statistickeskogo Analiza sluchainykh Protsessov s Pomoshch'yu Etsvm)

By

G.F. Telenin and F.V. Podsypanin

Moscow, 1970

Translated for the National Aeronautics and space Administration and the National Science Foundation, Washington, D.C. by Franklin Book Programs Inc., Cairo, 1978.

Contents


| | Page |
|---|---|
| Introduction..... | 3 |
| Chapter I. Spectral and Correlative Numerical Analyses of stationary Random Processes | 6 |
| §1. Formulation of the problem and justification of the method..... | 6 |
| §2. Construction of a model random process and its realization by a computer..... | 22 |
| §3. Derivation of the basic computation formulae..... | 27 |
| §4. Some general aspects of the practical application of the method..... | 34 |
| §5. The results of application of the method of the study of model and real random processes..... | 43 |
| Conclusion..... | 61 |
| References..... | 61. |

The present work discusses the numerical methods developed for the estimation of the statistical characteristics of stationary and non-stationary random processes and their realization by means of computers. The collection describes also the numerical methods of spectral and correlative analysis of stationary random processes. An expression for the numerical estimation of these characteristics is derived and analyzed. The results of the analysis of model processes of known statistical characteristics are especially realized on a computer, and the real experimental processes are represented as an extensive graphical information. The practical applicability of the numerical methods presented here, which are recommended to the scientists and engineers, is the main purpose of the present collection.

Scientific transactions No. 6

Georgiy Feodorovich Telenin
and Fedor Vladimirovich Podsypanin

Edited by N.V. Pakhomova
Corrected by L.N. Pobedonostseva

---

Signed for publication on 22.12.1970. L - 107745
Format: 60X90 1/8. Volume 6 Typographical units. Edition of 250 copies.
Order No 350 Price : 30 kopeeks.
Printed as a resolution of the Editorial Board of the Institute.

---

Printed in the Rotary Press of the Institute of Mechanics, (MGU (Moscow State University).

## Introduction

Stationary random processes are dealt with in aerodynamics, radioengineering theory of communication, biology and economics, in calculating the loads of flying vehicles, in designing rescuing equipment for transport, as well as in many different branches of science and technology. Some examples of the stationary processes are: turbulent pulsations and pulsation due to bad stream-lining of bodies in stationary fluxes, noise in resistance at constant temperature, intensity of steady signals in a receiving system, dependence of the thickness of a weaving thread on length, --- etc.

A random process is called stationary if its average characteristics are invariant under displacements along the independent-variable axis. This idealized scheme is suitable for processes with macroscopic conditions remaining invariable for a relatively long period of time. In these processes, one or the other type of statistical equilibrium is established.

For statistical processes satisfying the necessary ergodic conditions, one can define the probability characteristics according to one or several realizations. We usually confine our consideration to the evaluation of the correlation function and the spectral density, which determine the most important parameters of the statistical process for practical application. The determination of these characteristics can be achieved by two different methods. The first method is based on the application of special analogue instruments, viz: correlators and spectral analyzers, and the second one, on the utilization of numerical methods and computers. Different combinations of these methods can be also used.

Detailed analysis of the random processes is at the present time available in the literature, but will not be considered here. As a result of the general spread of computers, one may doubt the validity of the second method and the possibility of its application in practice.

The present chapter is devoted to the numerical methods used for estimation of the correlation function and the spectral density of a single process by means of computers, as well as the mutual correlation function of two processes. The first section includes a brief account of the principles of the theory of stationary random processes, which are necessary for the present work, and introduces the basic methods used for the derivation of the parameters. Using the results obtained

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR



for normal random processes, a numerical method for the estimation of spectral density is selected. The second section considers a family of model stationary processes having different spectral properties, and the methods developed for their realization with the aid of computers. The third section introduces and analyzes the main calculation formulae for the estimation of the correlation function and the spectral density. The basic factors influencing the estimation accuracy are examined in the fourth section. Methods for the optimum selection of the discretization step $\Delta t$, the realization time T and the average range of frequency $\Delta f$, for wide - and narrow - band processes, are considered. The obtained equations are used for the construction of a computer program. The fifth section comprises the results of the systematic calculation of the values of the correlation function and the spectral density of a series of model stationary processes, which illustrate the applicability conditions of the method, the influence of the basic factors on the accuracy, the method of selection of the optimum parameters $\Delta t$, T and $\Delta f$, as well as a number of the general properties of the estimates. This section gives also an example for the utilization of the numerical methods in analysing the real random processes obtained in the experiments.

For the effective utilization of a computer, it should be supplied with an attachment for the discretization of information and automatic supply. In the presence of such an attachment, the computer becomes quite flexible as a universal device for the study of different problems related to the statistical processes. Some positive features of the methods, based on the utilization of computers, are emphasized. All the basic work is done on the computer by its personnel. Carrying out the investigations related to the analysis of the random processes does not require a specific expensive equipment nor training the personnel for its adjustment and exploitation. If it is required to get additional statistical characteristics for the process under investigation, one can easily modify the numerical methods while using the computer, but in this case additional equipment will be required and we need to develop and adjust the new devices in the instrumental analysis. As far as the accuracy is concerned, both the instrumental and the numerical analyses have common and special sources of error. Of particular importance is the fact that on the utilization of the numerical methods for the estimation of the spectral density, the parameters that determine the accuracy of the process can be continuously changed, and optimum values can be obtained for each individual section of the spectrum. This

permits to carry out various studies requiring a relatively high accuracy.

The theory of random processes is the subject of an extensive literature. In order to limit the volume of the present work, we shall not give references in the text to individual sources. A detailed survey of the theory of stationary random processes is given in [1,2,3,4,7] . The necessary information about the ergodic properties of the processes is given in [1,2,3,7] . The fundamentals of the theory of instrumental analysis supported by examples is given in [4] . The systematic explanation of the theory of spectral density and the estimation of the correlation function are given in [7] , while the introductory material concerning this subject is also given in [1,2,3] . The methods of realization of random processes of given properties by the aid of computers are reviewed in [6] . The necessary background of the Fourier integral theory can be adopted from ref. [5] .

## CHAPTER 1. SPECTRAL AND CORRELATIVE NUMERICAL ANALYSES OF STATIONARY RANDOM PROCESSES

### § 1. Formulation of the Problem and Justification of the Method.

The problem is to select and test the applicability of the numerical method for the determination of the spectral and correlative characteristics of stationary random processes of certain given properties by means of a computer, and their individual realization.

Capital letters (X,Y, --- etc.) are used for denoting the random processes (functions), and the corresponding small letters (x,y, --- etc.) are used for denoting the realization of these processes. The dash over the letters denotes the mean value over a set of realizations while the letter $\mathcal{E}$ denotes the averaging operator. For example, the first order moment and the second - order displacement moments will be given by

$$\left.\begin{aligned} \overline{X(t)} &= \mathcal{E}[x] = \int_{-\infty}^{+\infty} x \cdot w_1(x,t)\,dx, \\ \overline{X(t)\cdot X(t')} &= \mathcal{E}[x\cdot x'] = \int_{-\infty}^{+\infty} x\cdot x'\cdot w_2(x,t;x',t')\,dx\,dx', \end{aligned}\right\} \qquad (1.1)$$

where $w_1(x,t)$, $w_2(x,t;x',t')$ are the corresponding uni-dimensional probability densities of the process X (t).

By a strictly stationary, or narrow-mindedly stationary, sence we call the stationary processes X (t) when any of their probability characteristics are invariant under displacement along the t- axis. For such processes, all the uni-dimensional probability densities and the moments of any order depend not on the values of the arguments $t_o$, $t_1$, $t_2$, ..., $t_n$, themselves, but only on their differences $t_1-t_o$, $t_2-t_o$, ..., and $t_n-t_o$, ....

We are interested in those characteristics which determine the first- and second-order moments. We shall thus mainly consider a more wide class of processes - viz.: the processes which are generally stationary. These processes are defined as follows:

1. The first moment $m_x = \overline{X}$ (t) is independent of t and will, in what follows, be set equal to zero, i.e. we will consider primarily the intermediate processes.

2. The mixed second moment or the correlation function

$$K_X(\tau) = \overline{X(t) \cdot X(t')} = K_X(-\tau) \qquad (2.1)$$

depends only on the difference between the arguments $\tau = t' - t$, such that

$$|K_X(\tau)| \leq K_X(0). \qquad (3.1)$$

The correlation function is assumed to be uniform throughout the whole range of its definition, and to guarantee this it is sufficient to assume that it is constant at the point $\tau = 0$ $(K_X(0) < \infty)$.

Accordingly, the stationary processes are mainly processes for which the uni- and the two-dimensional probability densities, and therefore the first and second moments, are invariant under displacements along the t-axis. It is clear that the stationary processes, in a strict sense, are also stationary in a general sense but not vice versa. An exception to this rule are those practically important processes whose uni-dimensional densities are all Gaussian functions. For such processes the first moment together with the correlation function define all the uni-dimensional probability densities, and thus the definitions of finiteness, in the strict and general senses, will coincide.

We now give an account of the elements necessary for the general theory of the stationary processes. From the definition of the correlation function and its uniformity condition, it follows that

$$K_X(\tau) = \int_{-\infty}^{\infty} e^{i 2\pi f \tau} \, dF_X(f), \qquad (4.1)$$

where f is the frequency and

$F_X(f)$ is the spectral distribution function of the process;

($F_X(f)$ is a non-decaying function and $F_X(-\infty) = 0$, $F_X(\infty) = 1$).

In general,

$$F_X = F_{X_1} + F_{X_2} + F_{X_3}, \qquad (5.1)$$

where $F_{x_1}$ is a non- stationary uniform function having a derivative equal to zero almost everywhere (except at a set of points of a zero measure),

$F_{x_2}$ is a step function with discontinuities at a given or a certain number of points, and

$F_{x_3}$ is an absolutely uniform function.

The singular component of $F_{x_1}$ is absent in practical problems and is usually set equal to zero. We shall not also consider the spectral distribution function of the discrete components of $F_{x_2}$, because, on one hand, it is not possible to get their spectral function numerically, and on the other hand, they can be quite closely approached by considering narrow-band processes. Consequently:

$$F_x(f) = F_{x_3}(f) = \frac{1}{2}\int_{-\infty}^{f} W_x(\xi)\,d\xi, \qquad (6.1)$$

where the non -negative functions $W_x(f)$ defines the spectral density. In this case, eq. (4.1) can be written as

$$K_x(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} W_x(f)\, e^{i2\pi f\tau}\,df. \qquad (7.1)$$

The spectral density is defined by:

$$W_x(f) = \lim_{T\to\infty} \mathcal{E}\left[\frac{2|S_{xT}(f)|^2}{T}\right] = \lim_{T\to\infty} \mathcal{E}\left[\frac{2 S_{xT}(f) S^{*}_{xT}(f)}{T}\right] = \qquad (8.1)$$

$$= \lim_{T\to\infty} 2\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)\cdot K_x(\tau)\, e^{-i2\pi f\tau}\,d\tau,$$

where

$$S_{xT}(f) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-i2\pi f t}\,dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-i2\pi f t}\,dt \qquad (9.1)$$

is the Fourier transformation of the "cut-off" realization

$$x_T(t) = \begin{cases} x(t) & \text{at } |t| \le \frac{T}{2} \\ 0 & \text{at } |t| > \frac{T}{2} \end{cases} \qquad (10.1)$$

while $S^{*}_{xT}(f)$ is the complex conjugate of the function $S_{xT}(f)$.

If the Fourier transformation of the correlation function exists, then it follows from (8.1) that

$$W_x(f) = 2\int_{-\infty}^{\infty} K_x(\tau)\, e^{-i\,2\pi f\tau}\, d\tau \qquad (11.1)$$

in all the points where $W_x(f)$ is uniform. The prerequisite conditions for fulfilling (11.1) can be expressed in different ways in terms of the correlation functi n on the spectral density. The first prerequisite condition requeires that $W_x(f)$ is uniform and that its variation is confined in any given range. The second prerequisite condition requires the absolute integrability of $K_x(\tau)$ $\left(\int_{-\infty}^{\infty} |K_x(\tau)|\, d\tau < \infty\right)$; that the integral (11.1) is uniformly convergent, and that $W_x(f)$ is also uniform. Equations (7.1) and (11.1) are known as the Khinchin - Winer theorem which is fundamental in the practical application of the general theory of stationary random processes. Taking into consideration the parity of $K_x(\tau)$ and $W_x(f)$, they can be rewritten as

$$K_x(\tau) = \int_0^{\infty} W_x(f)\, \cos 2\pi f\tau\, df \qquad (12.1)$$

and

$$W_x(f) = 4\int_0^{\infty} K_x(\tau) \cos 2\pi f\tau\, d\tau \,. \qquad (13.1)$$

Now, let X(t) and Y(t) represent general stationary processes which are rigidly connected, i.e., their mutual correlation function

$$K_{xy}(\tau) = \overline{X(t)\cdot Y(t')} = K_{yx}(-\tau) \qquad (14.1)$$

also depends on the argument difference $\tau = t - t'$ (which is already uneven). In analogy with eq. (8.1) the mutual spectral density of these processes is defined as:

$$W_{xy}(f) = \lim_{T\to\infty} \mathcal{E}\left[\frac{2S_T(f)\,S_{yT}^{*}(f)}{T}\right] \qquad (15.1)$$

From the properties of the Fourier transformation it is clear that the spectral density is a complex function of the frequency:

$$W_{xy}(f) = Re\, W_{xy}(f) + i\, Im\, W_{xy}(f)$$

with an even real part. $R_e W_{xy}$ and an odd imaginary part $J_m W_{xy}$. The mutual correlation function and the spectral density are related to each other by the Khinchin-Winer equations:

$$\left.\begin{aligned} K_{xy}(\tau) &= \int_{-\infty}^{\infty} W_{xy}(f)\, e^{i2\pi f\tau}\, df = K_{yx}(-\tau), \\ W_{xy}(f) &= \int_{-\infty}^{\infty} K_{xy}(\tau)\, e^{-i2\pi f\tau}\, d\tau = W_{yx}^{*}(f) \end{aligned}\right\} \qquad (16.1)$$

Let us consider the stationary process X(t) + Y (t) which is equal to the sum of the two stationary and rigidly connected processes X (t) and Y (t). For this process we have:

$$\left.\begin{aligned} K_{x+y}(\tau) &= K_x(\tau) + K_y(\tau) + 2K_{xy}^{c}(\tau), \\ W_{x+y}(f) &= W_x(f) + W_y(f) + 2W_{xy}^{c}(f) \end{aligned}\right\}, \qquad (17.1)$$

where

$$\left.\begin{aligned} K_{xy}^{c}(\tau) &= \frac{K_{xy}(\tau) + K_{xy}(-\tau)}{2}, \\ W_{xy}^{c}(f) &= \frac{W_{xy}(f) + W_{xy}^{*}(f)}{2} = R_e W_{xy}(f) \end{aligned}\right\} \qquad (18.1)$$

are the total values of the mutual correlation function and the spectral density. Thus, in order to determine the resultant of the two processes in any system, we should know the values of $K_{xy}^{c}(\tau)$ and $W_{xy}^{c}(\tau)$ in addition to the statistical parameters $W_x$, $W_y$, $K_x$ and $K_y$ of both processes.

Since the sum X (t) + Y (t) is also a general stationary process, it then follows from (17.1) and the basic relation (4.1) that

$$K_{xy}^{c}(\tau) = \int_{-\infty}^{\infty} e^{i2\pi f\tau}\, d\psi_{xy}(f), \qquad (19.1)$$

where $\psi_{xy}(f)$ is an absolutely convergent function with a limited variation in the infinite range $(-\infty \div +\infty)$, which is, in the general case, not a distribution function (i.e., not a non-decaying limited function). From eq. (16.1), it is easy to see

that $K^c_{xy}$ and $W^c_{xy}$ satisfy the Khinchin-Winer equation:

$$\left.\begin{aligned} K^c_{xy}(\tau) &= \int_0^\infty W^c_{xy}(f)\cos 2\pi f\tau \, df, \\ W^c_{xy}(f) &= 4\int_0^\infty K^c_{xy}(\tau)\cos 2\pi f\tau \, d\tau \end{aligned}\right\} \qquad (20.1)$$

where $W^c_{xy}(f)$ is not, in general, a non-negative function.

Owing to the uniformity of the spectral distribution functions $F_x(f)$ and $F_y(f)$ as well as the function $\gamma_{xy}(f)$, it follows from eq. (4.1) and (5.1) that

$$\left.\begin{aligned} &\lim_{T\to\infty}\frac{1}{T}\int_0^T K_x(\tau)d\tau = \lim_{T\to\infty}\frac{1}{T}\int_0^T K_y(\tau)d\tau = 0, \\ &\lim_{T\to\infty}\frac{1}{T}\int_0^T K_x^2(\tau)d\tau = \lim_{T\to\infty}\frac{1}{T}\int_0^T K_y^2(\tau)\,d\tau = 0, \\ &\lim_{T\to\infty}\frac{1}{T}\int_0^T K^c_{xy}(\tau)d\tau = \lim_{T\to\infty}\frac{1}{T}\int_0^T [K^c_{xy}(\tau)]^2 d\tau = 0 \end{aligned}\right\} \qquad (21.1)$$

It turns out that the fulfillment of eq. (21.1) for $K_x^2$, $K_y^2$ and $[K_{xy}]^2$ is both a necessary and a prerequisite condition for the uniformity of $F_x$, $F_y$ and $\gamma_{xy}$.

We shall now introduce the random functions $Q_{xT}$ and $R_{xT}$, whose expressions are given by:

$$\left.\begin{aligned} Q_{xT} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt, \\ R_{xT} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cdot x(t+\tau)\,dt \end{aligned}\right\}. \qquad (22.1)$$

Averaging the left- and right- band sides of eq. (22.1) over the realization set, and noting that $m_x=0$ and that $K_x$ is independent of the integration variable, which can thus be taken outside the integration sign, we get:

$$\mathcal{E}[q_{xT}]=0,\ \mathcal{E}[r_{xT}]=K_x(\tau)\ , \tag{23.1}$$

i.e., $q_{xT}$ and $r_{xT}$ give the unmixed statistical estimates of $m_x$ and $K_x(\tau)$ for any given value of T. In what follows, we will be interested in the choice of a numerical method for the estimation of the spectral density $W_x(f)$ and the correlation function $K_x(f)$ by a single parameter $x(t)$. Therefore, it is natural to assume that the random processes under consideration are ergodic with respect to the first moment and the correlation function. Then, on the average, the following relations of convergence hold

$Q_{xT} \to m_x=0$ и $R_{xT} \to K_x(\tau)$ $T\to\infty$, т.е.

when $T \to \infty$, i.e. when

$$\begin{aligned} &\lim_{T\to\infty} \mathcal{E}[q_{xT}^2]=0, \\ &\lim_{T\to\infty} \mathcal{E}[|r_{xT}-K_x(\tau)|^2]=0 \end{aligned} \tag{24.1}$$

Accordingly, for the ergodic processes, $q_{xT}$ and $r_{xT}$ are the stationary estimates of $m_x$ and $K_x(\tau)$ (whose distribution tends to zero as $T \to \infty$) in addition to the undisplaced estimate. Using Chebyshev's inequality for the random variables we have:

$$P\{|\lambda-\alpha|>\varepsilon\} \le \frac{|\lambda-\alpha|^2}{\varepsilon^2},\ (\varepsilon>0)\ , \tag{25.1}$$

where P denotes the probability of satisfying the relationship in the brace br ts; it is easy to see that eq. (24.1) implies the probable convergence of the estimates

$$\left.\begin{aligned} &\lim_{T\to\infty} P\{|Q_{xT}|>\varepsilon\}=0 \\ &\lim_{T\to\infty} P\{|R_{xT}-K_x(\tau)|>\varepsilon\}=0 \end{aligned}\right\}. \tag{26.1}$$

The last relationship shows that the probability of variation of the estimates $q_{x\,T}$ and $r_{x\,T}$ from $m_x$ and $K_x$ by a very small definite value tends to zero as $T \to \infty$. In other words, as $T \to \infty$, the probability densities of these estimates tend to the delta-functions $\delta(q_{x\,T}$ and $\delta(r_{x\,T} - K_x(\tau))$

We shall now joint out the limitations implied by the ergodic condition on the processes under consideration. In the theory of stationary processes, it is shown that for the fulfilment of the first relationship of (24.1), which implies ergodicity relative to the first moment $m_x$, it is necessary and prerequisite that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K_x(\tau)\,d\tau = 0, \tag{27.1}$$

But, this relationship (see eq. 21.1) is always satisfied for the processes considered here which have a uniform spectral distribution function. The situation is more difficult when the second relationship of (24.1), which implies ergodicity relative to the correlation function, is considered. The following intermediate random process has been considered:

$$Y = X(t)\cdot X(t+\varepsilon) - K_x(\tau), \tag{28.1}$$

where $\tau$ is a given parameter. If the general stationary initial process X (t) is such that its fourth moment is limited and invariant under displacement in t (see the beginning of this section), then the process Y (t) will also be in general stationary because

$$\overline{X(t)\cdot X(t+\tau)\cdot X(t+\sigma)\cdot X(t+\sigma+\varepsilon)} - K_x^2(\tau) = \overline{Y(t)\cdot Y(t+\sigma)} = K_Y(\tau,\sigma) \tag{29.1}$$

and its correlation function will depend only on $\sigma$ and not on t. Consequently (see 27.1), the following equation

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K_Y(\tau,\sigma)\,d\sigma = 0 \tag{30.1}$$

is a necessary and prerequisite condition for the ergodicity of the process Y(t) relative to the first moment and, according to (24.1), for the ergodicity of the initial process X(t) relative to the correlation function, because

$$q_{Y\,T} = r_{x\,T} - K_x(\tau). \tag{31.1}$$

If the relationship between the values of the random functions X(t) and Y(t) (see 28.1) at the two points $t=t_1$ and $t=t_2$ infinitely decreases as $(t_2-t_1) \to \infty$, then

$$\lim_{\tau \to \infty} K_x(\tau) = 0 \qquad (32.1)$$

$$\lim_{\sigma \to \infty} K_Y(\tau,\sigma) = 0 \qquad (33.1)$$

It is sufficient to fulfil these relationships for the ergodicity of the processes under consideration because they clearly lead to eq. (27.1) and (30.1).

If the general stationary process X(t) is normal, then (see the beginning of this section) it is stationary in the strict sense point of view, i.e. all its moments are invariant under displacement in t. It is easy to show that in the case of a normal process, condition (30.1) is equivalent to the following relationship:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T [K_x(\sigma)]^2 \, d\sigma = 0 \quad , \qquad (34.1)$$

which is, according to (21.1), necessary and prerequisite for the uniformity of the spectral distribution function $F_x(f)$.

Moreover, for a normal process, ondition (34.1) is necessary and prerquisite for an ergodicity relative to moments of any order. Thus, any stationary normal process is ergodic if it has a uniform spectral distribution function. For a normal process, it is clear that equation (32.1) is a prerequisite condition for ergodicity because it leads to eq. (34.1).

From the above arguments it is clear that the property of ergodicity is closely related to the uniformity of the spectral distribution function or, respectively, to the character of the decay of the correlation functions. These properties are satisfied by a wide class of staionary processes which are often delt with in practice.

In considering the mutual correlation functions and the spectral densities of the two stationary and rigidly connected processes X(t) and Y(t), it is also necessary to consider the ergodicity of these processes relative to the mutual correlation function (14.1). In this case, it is easy to carry out

an analogous analysis for the ergodicity conditions.

As mentioned above, the parameter $r_{xT}(\tau)$ (see 22.1) gives an undisplaced and consistent estimate of the correlation function $K_x(\tau)$. The main difficulty is in estimating the spectral density $W_x(f)$. In order to justify the numerical method for the calculation of $W_x(f)$, we shall use some of the results obtained for the normal processes since the real processes are in many cases close to them. The verification of the conclusions of these results, as well as the qualitative and quantitative study of the problems met with on the practical application of the method, will be carried out by means of the non-normal stationary model process X(t) which is realized by means of a computer (see § 2 and 5) and which has the known functions $K_x(\tau)$ and $W_x(f)$. In addition, the method will be applied to the analysis of the values of the real processes obtained in the experiments.

We shall now set up a statistical estimate for the spectral density $W_x(f)$ by the realization of the general stationary process X(t), which is ergodic relative to the first moment and the correlation function. Since, in practice, the process can be realized only in a given period of time of a value T, we should then take into consideration the truncated process $X_T(t)$ having a value of $x_T(t)$ defined by (10.1). According to (8.1), the non-negative function:

$$\varphi_{xT}(f) = \frac{2\,|S_{xT}(f)|^2}{T} = \frac{2\,S_{xT}(f)\,S_{xT}^{*}(f)}{T}, \qquad (35.1)$$

which is normally called a periodic function, is an asymptotically undisplaced estimate of the spectral function $W_x(f)$. It appears, however, that the distribution of the random function $\varphi_{xT}$ (with the value $\varphi_{xT}$) does not tend to zero as $T \to \infty$ for a wide class of processes. For example, for a normal process we have:

$$\lim_{T\to\infty} E\left[\left|\varphi_{xT} - E[\varphi_{xT}]\right|^2\right] = W_x^2(f) \qquad (36.1)$$

Consequently, the parameter $\varphi_{xT}$ is in consistant and is therefore of no practical value. It would be shown that as T increases, the random function $S_{xT}(f)$ (with a value of $S_{xT}(f)$, see 9.1) converges to the complex random function of the frequency V(f), which has a correlation equal to $W_x(f) \cdot \delta(f-f')$. This random function is called the white noise. Its values are

uncorrelated at arbitrary two points and are infinitely distributed. Thus, in the limit of $T \to \infty$, the function $\varphi_{xT}(f)$ has an infinitely large number of vibrations in any arbitrary small range of frequency. As an example, fig. 1 and 2 show a number of periodograms, calculated for wide- (fig. 1) and narrow- band real random processes of pressure fluctuations. By comparing the curves, it is noted that the frequency of the periodograms increases with the value of the realization time; the "period" of the f-vibration is of the order of $\frac{2}{T}$. It thus follows that, for a fixed frequency, when $T \to \infty$, the parameter $\varphi_{xT}(f)$ does not tend to a limiting value, whereas for a frequent change of T we get one vibration.

The considered parameters of the periodograms imply that their average characteristics in an infinitesimally small range of frequency depending on T, might give a satisfactory value for the spectral function, which is displaced but has a decaying distribution as $T \to \infty$.

$$\tilde{\varphi}_{xT}(f) = \int_{-\infty}^{+\infty} \delta_T(\xi - f)\, \varphi_{xT}(\xi)\, d\xi , \qquad (37.1)$$

where $\delta_T(\xi) \geq 0$ is almost an arbitrary weight function having a sharp peak at $\xi = 0$ and small values elsewhere, satisfying the condition:

$$\int_{-\infty}^{+\infty} \delta_T(\xi)\, d\xi = 1 \qquad (38.1)$$

and tending to a $\delta$-function as $T \to \infty$. A special case is the following "rectangular" weight function:

$$\delta_T(\xi) = \begin{cases} \frac{1}{\Delta f_x} & \text{when } f - \frac{\Delta f_x}{2} \leq \xi \leq f + \frac{\Delta f_x}{2}, \\ 0 & \text{when } \xi < f - \frac{\Delta f_x}{2} \text{ and } \xi > f + \frac{\Delta f_x}{2}. \end{cases} \qquad (39.1)$$

Equation (37.1) will then take the following form:

$$\tilde{\varphi}_{xT}(f) = \frac{1}{\Delta f_x} \int_{f - \frac{\Delta f_x}{2}}^{f + \frac{\Delta f_x}{2}} \varphi_{xT}(\xi)\, d\xi , \qquad (40.1)$$

where $\Delta f_x$ must decrease according to a given rule on the

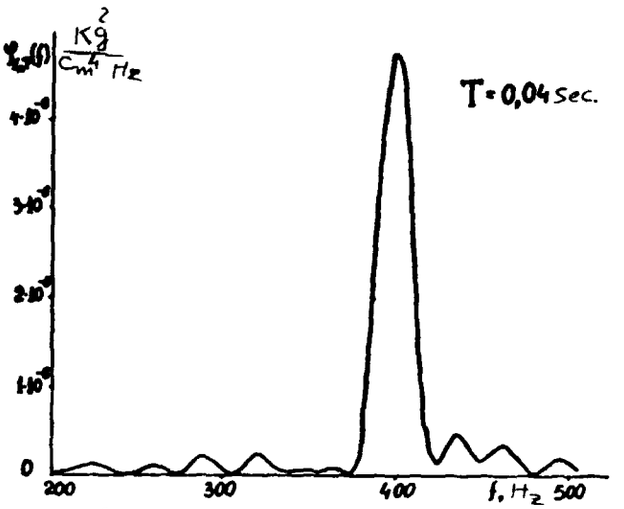

T = 0,04 sec.
T = 0,16 sec.
200
300
400
500
f, Hz

Fig. 1

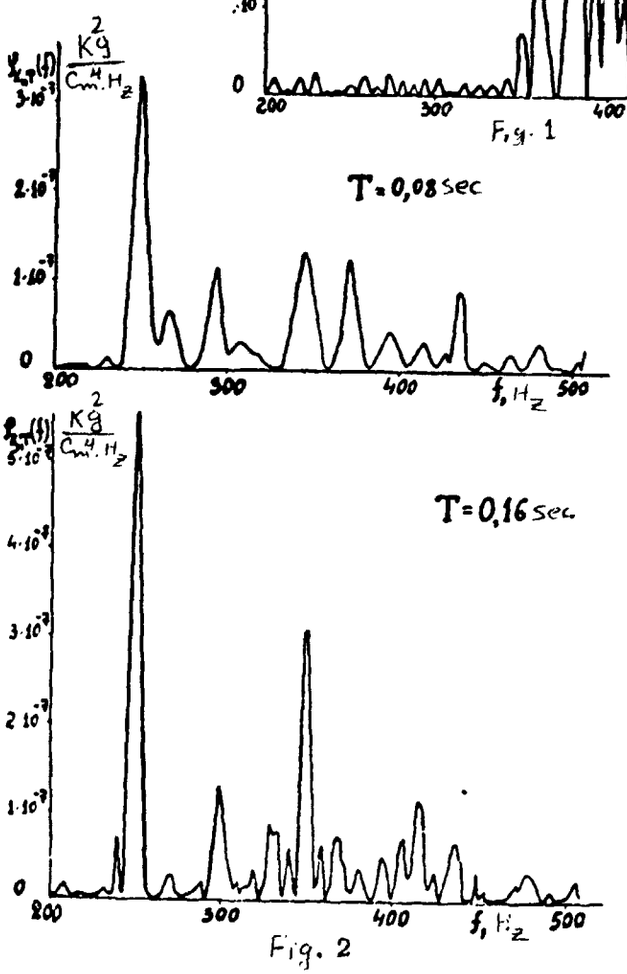

T = 0,08 sec
T = 0,16 sec
200
300
400
500
f, Hz

Fig. 2

decrease of T. For normal processes, there is a general theory for such estimates.

Since the application of the numerical method implies that the parameter $x_n(t)$ is realized in discrete points with a step $\Delta t$, i.e. the random process X(t) is replaced by the random sequence $X_n = X(n \cdot \Delta t)$, , an account of the results of this theory concerning sequences will be thus given. Let us assume that in N points ($t = t_n$, $n = 1,2,\ldots, N$) of the interval $-\frac{T}{2} \leq t \leq \frac{T}{2}$ with a step $\Delta t = \frac{T}{N}$, the values $x_n = x(n \cdot \Delta t)$ of the parameter are given. Replacing the integration by a summation, using the simplest quadrature formula of the rectangles, we get from (1.1), (8.1) , (9.1) , (22.1) and (35.1) the following

$$K_{x_n}(m\Delta t) = \mathcal{E}[x_n \cdot x_{n+m}] = K_x(m \cdot \Delta t) \qquad (m = 0,1,\ldots,N-1), \tag{41.1}$$

$$W_{x_n}(f) = \lim_{N\to\infty} \mathcal{E}\left[\frac{2 S_{x_n T}(f) \cdot S^{*}_{x_n T}(f)}{N \cdot \Delta t}\right], \tag{42.1}$$

$$V_{x_n T}(m \cdot \Delta t) = \frac{1}{N - m} \sum_{n=1}^{N-m} x_n \cdot x_{n+m} \qquad (m = 0,1,\ldots,N-1); \tag{43.1}$$

$$\gamma_{x_n T}(f) = \frac{2 S_{x_n T}(f) \cdot S^{*}_{x_n T}(f)}{N \cdot \Delta t}, \tag{44.1}$$

where

$$S_{x_n T}(f) = \Delta t \sum_{n=1}^{N} x_n \, e^{-i 2\pi f (n - N/2) \Delta t}. \tag{45.1}$$

From these relationships it is clear that the parameter $\gamma_{x_n T}(m \cdot \Delta t)$ Is undisplaced and that the frequency functions $S_{x_n T}$, $W_{x_n}$ and $\gamma_{x_n T}$ are periodical functions with an amplitude of $\frac{1}{\Delta t}$. When $\Delta t \to 0$, the sum of the integral converges to the corresponding integral, and the statistical characteristics of the uniform and discrete values of the parameter tend to have identical results.

Although equation (40.1) is not optimum, yet it is favourable in that its derived value has a simple physical and mathematical meaning. After clarifying the basic question of the practical determination of the estimates, the application of more general weight functions $E = \varepsilon$, becomes straight forward.

In the method used for estimation of normal stationary processes having uniform spectral densities with two uniform derivatives, the following basic asymptotic relationships are obtained:

$$\mathcal{E}\left[\left|\hat{K}_{x_n T}(m\Delta t) - K_x(m\Delta t)\right|^2\right] \sim \frac{1}{4(N-m)\Delta t}\int_{-\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} K_{x_n}^2(f)\,df \times \left[1 + \cos(4\pi f m\Delta t)\right]df, \tag{46.1}$$

$$\mathcal{E}\left[\tilde{\varphi}_{x_n T}(f)\right] \sim \frac{1}{\Delta f_x}\int_{f-\frac{\Delta f_x}{2}}^{f+\frac{\Delta f_x}{2}} W_{x_n}(\xi)\,d\xi, \tag{47.1}$$

$$\mathcal{E}\left[\left|\tilde{\varphi}_{x_n T}(f) - \mathcal{E}\left[\tilde{\varphi}_{x_n T}(f)\right]\right|^2\right] \sim \frac{1}{N\cdot\Delta t\cdot\Delta f_x}\cdot\frac{1}{\Delta f_x}\int_{f-\frac{\Delta f_x}{2}}^{f+\frac{\Delta f_x}{2}} W_{x_n}^2(\xi)\,d\xi, \tag{48.1}$$

where

$$\tilde{\varphi}_{x_n T}(f) = \frac{1}{\Delta f_x}\int_{f-\frac{\Delta f_x}{2}}^{f+\frac{\Delta f_x}{2}} \varphi_{x_n T}(\xi)\,d\xi \tag{49.1}$$

is the discrete analogue of eq. (40.1). In writing eq. (46.1), it is considered that:

$$\mathcal{E}\left[\hat{K}_{x_n T}(m\Delta t)\right] = K_{x_n}(m\Delta t) \sim K_x(m\Delta t),$$

because of the non-displacement of the parameter $\hat{\gamma}_{x_n T}$ and (41.1)

From eq. (46.1) it is clear that for any fixed value of $m\Delta t$, the distribution of the parameter $\hat{\gamma}_{x_n T}$ can be made very small. However, for this purpose, the value must be much greater than $m\Delta t$.

Equations (47.1) and (48.1) show that, when $N\Delta t \to \infty$, the parameter $\tilde{\varphi}_{x_n T}(f)$ gives an undisplaced and consistent value for the spectral density, defined in the interval $\left(f - \frac{\Delta f_x}{2}, f + \frac{\Delta f_x}{2}\right)$. For the spectral density $W_{x_n}(f)$ itself, this value is approximately displaced at $B\Delta T$. The asymptotic expression for this displacement is given by:

$$\left.\begin{aligned}\Delta_f &\sim \frac{1}{\Delta f_x}\int_{f-\frac{\Delta f_x}{2}}^{f+\frac{\Delta f_x}{2}} W_{x_n}(\xi)\,d\xi - W_{x_n}(f) = \\ &= \frac{1}{\Delta f_x}\int_{f-\frac{\Delta f_x}{2}}^{f+\frac{\Delta f_x}{2}}\left[W_{x_n}(\xi) - W_{x_n}(f)\right]d\xi\end{aligned}\right\} \tag{50.1}$$

In deriving the basic asymptotic equations (47.1) and (48.1), it is assumed that a uniform second derivative $W''_{x_n}(\xi)$ exists. Putting $\xi = f + \ell$, we get:

$$W_{x_n}(\xi) = W_{x_n}(f) + W'_{x_n}(f)\ell + \frac{1}{2}W''_{x_n}(f+\theta\ell)\ell^2,$$

where $0 < \theta < 1$. Substituting this expression in eq. (50.1) and using the generalized averaging theorem for the integrals, we get:

$$\Delta_1 \sim \frac{W'_{x_n}(\xi_1)}{2N}\cdot\Delta f_x^2, \quad f - \frac{\Delta f_x}{2} < \xi_1 < f + \frac{\Delta f_x}{2}. \qquad (51.1)$$

Similarly, for the mean square deviation of the parameter $f_{x_n T}$

$$\Delta_2 = \sqrt{\mathcal{E}\left[\left|\gamma^0_{x_n T} - \mathcal{E}\left[\gamma^0_{x_n T}\right]\right|^2\right]}$$

we get the following equation according to (48.1), when $N\Delta t \to \infty$

$$\Delta_2 \sim \frac{1}{\sqrt{N\Delta t\cdot\Delta f_x}} W_{x_n}(\xi_2), \quad f - \frac{\Delta f_x}{2} < \xi_2 < f + \frac{\Delta f_x}{2}. \qquad (52.1)$$

From equations (51.1) and (52.1) it is clear that a certain relationship holds between $\Delta_1$ and $\Delta_2$. At a fixed value of $N\cdot\Delta t$, for the average range of frequency $\Delta f_x$, the displacement $|\Delta_1|$ decreases while the mean square deviation $\Delta_2$ increases. On the contrary, when $\Delta f_x$ increases, $|\Delta_1|$ increase and $\Delta_2$ decreases. We shall now find the value of $\Delta f_x$ that makes $\Delta = |\Delta_1| + \Delta_2$ take a minimum value. From the condition $\frac{d\Delta}{d\Delta f_x} = 0$ at large values of $N\Delta t$, it follows that:

$$N\Delta t\,\Delta f_x = \left[6\cdot\frac{W_{x_n}(\xi_2)}{|W''_{x_n}(\xi_1)|}\right]^{2/5}(N\cdot\Delta t)^{4/5} \qquad (53a.1)$$

or

$$\Delta f_x = \left[6\cdot\frac{W_{x_n}(\xi_2)}{|W''_{x_n}(\xi_1)|}\right]^{2/5}\cdot\frac{1}{(N\cdot\Delta t)^{1/5}}. \qquad (53b.1)$$

It is noted that the condition (53.1) also agrees with natural requirement of such a relationship between $N\Delta t$ and $\Delta f_x$, so that $|\Delta_1|$ and $\Delta_2$ are always varied rleative to each other:

$$\frac{|\Delta_1|}{\Delta_2} = A = const.$$

eq. (53a.1) corresponds to A=1/4.

Consequently, when the average range of frequency $\Delta f_x$ decreases as $\frac{1}{(N\Delta t)^{\gamma_5}}$, by increasing the product $N\Delta t$, the displacement $N\Delta t$ and the distribution factor $\Delta_2$ will then decrease. In other words, $\tilde{\varphi}_{x_n T}(f)$ gives an asymptotically undisplaced and consistent value for the spectral density $W_{x_n}(f)$, which needs to be verified. According to (53.1), when $N\Delta t$ increases, the parameter $\Delta f_x$ must decrease so slowly that the product $N\ \Delta t\ \Delta f_x$ will increase almost proportionally to $N\ \Delta t$. Since the amplitude of the periodogram is proportional to $\frac{2}{N\Delta t}$, then in the limit of $N\Delta t \to \infty$, the number of the random vibrations of the periodogram in the average range of frequency will also tend to infinity.

Equations (47.1) and (48.1), as well as eq. (53.1), are obtained for a special, though practically general, type of processes. Most probably, the qualitative conclusions are correct and eq. (53.1) is approximately valid for most of the ergodic processes.

By a suitable selection of the discretization step $\Delta t$, the following relationships can be valid to within a very high accuracy:

$$\left.\begin{aligned} \bar{W}_{x_n}(f) &\simeq W_x(f), \\ \tilde{\varphi}_{x_n T}(f) &\simeq \tilde{\varphi}_{xT}(f). \end{aligned}\right\}. \qquad (54.1)$$

All the previous arguments are completely related to the parameter $\tilde{\varphi}_{xT}(f)$ of the spectral density $W_x(f)$ of the initial process $X(t)$. In applying eq. (53.1), we must substitute $T=N\Delta T$ and take eq. (54.1) into consideration.

Real random processes are never exactly stationary, even in a general sense. In solving practical problems, we usually ask: when can we apply the theory of stationary processes? It is natural to consider the process stationary from a practical view point when the external conditions that determine its progress remain stable during a time interval $T_{min}$. As we have already seen, the required minimum value of the realization time $T_{min}$ always exists for any given degree of accuracy of the spectral density.

## § 2. Construction of a Model Random Process and its Realization by a Computer

We shall now consider the construction of the stationary and rigidly connected model processes X(t) and Y(t), which are characterized by known correlation functions ($K_x(\tau)$ and $K_y(\tau)$ and spectral densities ($W_x(f)$ and $W_y(f)$ as well as mutual correlation functions ($K_{xy}(\tau)$ and $W_{xy}(f)$). In this case, it is necessary that the process X(t) and Y(t) should be ergodic with respect to the first moment, the correlation functions and the mutual correlation functions, and that they should be easily realized by a computer. As X(t), we take the simplest random process.

$$X(t) = \cos[2\pi f_H t + G(t)], \quad 0 \le G \le 2\pi, \qquad (1.2)$$

where $f_H$ is a fixed carrier frequency, and G(t) is a random function with a value of $g(t)$.

In order to be able to construct the function from point to point, we take as G(t) the well-known uniform Markoff process, which has the following transition probability:

$$\nu(g_\tau/\tau, g) = \frac{1}{2\pi}\left\{1 + 2\sum_{j=1}^{\infty} e^{-j^2 d\tau}\cos j(g_\tau - g)\right\}, \qquad (2.2)$$

where $g_\tau = g(t+\tau)$ and $d > 0$. When $\tau \to \infty$, the conditional probability density $g_\tau$ relative to $g$, which is equal to $\nu(g_\tau/\tau, g)$, is independent of $g$, and thus becomes equal to the uniform probability density, i.e.

$$w_1(g) = \lim_{\tau\to\infty} \nu(g_\tau/\tau, g) = \frac{1}{2\pi}, \quad 0 \le g \le 2\pi. \qquad (3.2)$$

Since the uniform probability density is independent of t, then the arbitrary n-dimensional probability density:

$$w_n(g_1, t_1; g_2, t_2; \ldots; g_n, t_n) = w_1(g_1)\cdot\nu(g_2/\tau_{12}, g_1)\cdot\nu(g_3/\tau_{23}, g_2)\cdots\nu(g_n/\tau_{n-1,n}, g_{n-1})$$

will depend only on the argument increments $\tau_{12}, \tau_{23}, \ldots, \tau_{n-1,n}$, and the stationary process G(t) will be strictly stationary (see § 1). Since G(t) is uniformly distributed in the interval $[0, 2\pi]$ (3.2), then the process X(t) will be also strictly stationary. To calculate the correlation function and the

spectral density of this process, the simplest trigonometric relations are used to get:

$$m_x = \mathcal{E}[x(t)] = \cos 2\pi f_N t \int_0^{2\pi} \cos\varphi\, w_1(\varphi)\, d\varphi - \sin 2\pi f_N t \int_0^{2\pi} \sin\varphi\, w_1(\varphi)\, d\varphi = 0, \tag{4.2}$$

$$K_x(t,\tau) = \mathcal{E}[x(t)\cdot x(t+\tau)] = \int_0^{2\pi}\int_0^{2\pi} U_2(t,\tau;\varphi,\varphi_\tau)\cdot w_2(\varphi,\ \ -)\, d\varphi\, d\varphi_\tau ;$$

$$U_2(t,\tau;\varphi,\varphi_\tau) = \frac{1}{2}\{\cos 2\pi f_N \tau\cdot\cos(\varphi_\tau-\varphi) - \sin 2\pi f_N \tau\cdot\sin(\varphi_\tau-\varphi) + \cos 2\pi f_N(2t+\tau)\times \cos(\varphi_\tau+\varphi) - \sin 2\pi f_N(2t+\tau)\cdot\sin(\varphi_\tau+\varphi)\}, \tag{5.2}$$

$$w_2(\varphi,\tau,\varphi_\tau) = w_1(\varphi)\cdot v(\varphi_\tau|\tau,\varphi) = \frac{1}{2\pi} v(\varphi_\tau|\tau,\varphi)$$

From the values of the parameters $W_1$ and $v$ (see 2.2 and 3.2) it follows that

$$\mathcal{E}[\cos(\varphi_\tau+\varphi)] = \mathcal{E}[\sin(\varphi_\tau+\varphi)] = \mathcal{E}[\sin(\varphi_\tau-\varphi)]\cdot\mathcal{E}[\cos\varphi] = \mathcal{E}[\sin\varphi] = 0.$$

Substracting therefore $\mathcal{E}[\cos(\varphi_\tau-\varphi)]$, we get:

$$m_x = 0$$
$$K_x(\tau) = \frac{1}{2} e^{-\alpha|\tau|}\cdot\cos 2\pi f_N \tau. \tag{6.2}$$

As expected, the dependence of $m_x$ and $K_x$ on t disappears. Using the Khinchin Winer theorem (see eq. 11.1), we get the following expression for the spectral density:

$$W_x(f) = \alpha\left\{\frac{1}{4\pi^2(f-f_N)^2+\alpha^2} + \frac{1}{4\pi^2(f+f_N)^2+\alpha^2}\right\}. \tag{7.2}$$

A random process is called narrow-banded if its spectral density is localized in the vicinity of the fixed frequency $f_0$ within the following relatively narrow range (band) of frequency:

$$\left(f_0 - \frac{\Delta f_0}{2},\ f_0 + \frac{\Delta f_0}{2}\right), \text{ i.e. } \Delta f_0 \ll f_0 \qquad (8.2)$$

If this condition is not fulfilled, the process is then called wide-banded. Thus, the small value of d corresponds to a narrow- band model and its large value corresponds to a wide-band one. We shall now consider the intermediate random process.

$$Z(t,\tau) = X(t)X(t+\tau) - K_x(\tau),$$

where $\tau$ is a given parameter. The correlation function of this process is given by:

$$\left.\begin{aligned} K_z(t,t+\sigma,\tau) &= \mathcal{E}[Z(t,\tau)\cdot Z(t+\sigma,\tau)] = \\ &= \int_0^{2\tau}\int_0^{2\tau}\int_0^{2\tau}\int_0^{2\tau} U_4(t,\tau,\sigma;g,g_\tau,g_\sigma,g_{\sigma+\tau})\, w_4(g,\tau,g_\tau, \\ &\quad \sigma-\tau,g_\sigma,\tau,g_{\sigma+\tau})\, dg\, dg_\tau\, dg_\sigma\, dg_{\sigma+\tau} - \\ &\quad - K_x^2(\tau), \\ U_4(t,\tau,\sigma;g,g_\tau,g_\sigma,g_{\sigma+\tau}) &= U_2(t,\tau;g,g_\tau)\cdot U_2(t+\sigma,\tau;g_\sigma,g_{\sigma+\tau}) \\ w_4(g,\tau,g_\tau,\sigma-\tau,g_\sigma,\tau,g_{\sigma+\tau}) &= \frac{1}{2\pi}\, v(g_\tau/\tau,g)\cdot \\ &\quad \times v(g_\sigma/\sigma-\tau,g_\tau)\cdot v(g_{\sigma+\tau}/\tau,g_\sigma). \end{aligned}\right\} \quad (8.2)$$

Since the initial process is strictly stationary, then the dependence on t will disappear, and $K_z$ will depend only on $\sigma$ and $\tau$ (see §1).

For a fixed value of $\tau$, it follows from eq. (2.2) that

$\tau \lim_{\sigma\to\infty} \nu(\alpha,\beta-\tau;\sigma,\tau)=\frac{1}{2\pi}$. Using this property, we get from equations (5.2), (6.2) and (9.2):

$$\left.\begin{aligned} &\lim_{\tau\to\infty} K_x(\tau)=0, \\ &\lim_{\sigma\to\infty} K_z(\sigma,\tau)=K_x^2(\tau)-K_x^2(\tau)=0 \end{aligned}\right\}. \qquad (10.2)$$

According to equations (32.1) and (33.1), these relationships show the ergodicity of the present model stationary process X(t) relative to the first moment and the correlation function (see 1.2, 2.2 and 3.2).

To study the method of determination of the mutual correlation function and the spectral density, we need to consider the other model process Y(t). For its derivation, the simplest method is to apply a linear transformation to the process X(t) because this conserves the rigidity and ergodicity properties and does not significantly increase the calculation time. One of the simplest transformations is

$$Y(t)=\frac{X(t+t_*)-X(t)}{2}, \qquad (11.2)$$

where $t_*$ is a given parameter. The characteristics of this process are easily expressed in terms of those of the process X(t). Using the definition of $m_x$ and $K_x$, we get:

$$\left.\begin{aligned} &m_Y=0 \\ &K_Y(\tau)=\frac{2K_x(\tau)-K_x(\tau+t_*)-K_x(\tau-t_*)}{4}, \\ &K_{XY}(\tau)=\frac{K_x(t_*+\tau)-K_x(\tau)}{2}, \\ &K_{YX}(\tau)=\frac{K_x(\tau-t_*)-K_x(\tau)}{2} \end{aligned}\right\}. \qquad (12.2)$$

The last two equations of the series (12.2) and definition (11.2) show that the processes X(t) and Y(t) are rigidly connected and ergodic relative to the mutual correlation function. Substituting the second and third equations of the series (12.2) into the Khinchin - Winer equations (13.1) and (16.1), the spectral densities will be given by:

$$\left.\begin{aligned} W_y(f) &= \sin^2(\pi f t_n)\, W_x(f), \\ W_{xy}(f) &= \tfrac{1}{2}\left(e^{i2\pi f t_n} - 1\right) W_x(f), \\ \mathrm{Re}\, W_{xy}(f) &= -\sin^2(\pi f t_n)\, W_x(f), \\ \mathrm{Im}\, W_{xy}(f) &= \tfrac{1}{2}\sin(2\pi f t_n)\, W_x(f) \end{aligned}\right\}. \qquad (13.2)$$

By using more general linear transformations, we may clearly get pairs of processes having various propoerties of the mutual correlation function and the mutual spectral density.

We shall now briefly discuss the computer realization of the model stationary process (1.2). Modern computers are usually supplied by generators of pseudorandom numbers $\nu$, uniformly distributed in the interval $[\,0.1\,]$. In the present case, in order to determine the random quantities $\mu$ of a given distribution $\nu(\mu)$, it is convenient to use the following relationship :

$$\rho = \int_0^{\mu} \nu(\xi)\, d\xi. \qquad (14.2)$$

The values of realization time of the random prccess (1.2) are written at the discrete points $t=t_n$, where $n= 1,2,3,\ldots, N$ are expressed in the form

$$\left.\begin{aligned} x_n &= \cos\left[2\pi f_n (n - N/2)\Delta t + \vartheta_n\right], \\ \vartheta_n &= \vartheta_1 + \sum_{k=1}^{n-1} \gamma_k \end{aligned}\right\}, \qquad (15.2)$$

Here $\Delta t = \frac{T}{N}$ is the discretization step, and the parameter $\vartheta$ (see 3.2) is uniformly distributed over the interval $[0, 2\pi]$, whereas the parameters $\nu_k$ (see 2.2) are distributed according to

$$\nu(\nu_k) = \frac{1}{2\pi}\left\{1 + 2\sum_{j=1}^{\infty} e^{-j^2\alpha\Delta t}\cos j\nu_k\right\}. \quad (16.2)$$

By means of the generator of the uniformly distributed parameters, we first call the number $\rho_1$, and, according to (15.2),

$$\left.\begin{aligned} x_1 &= \cos\left[2\pi f_N(1 - N/2)\Delta t + \vartheta_1\right], \\ \vartheta_1 &= 2\pi\rho_1 . \end{aligned}\right\} \quad (17.2)$$

Then, we call the second number $\rho_2$. Substituting eq. (16.2) for the value of $\nu_1$ into eq. (14.2), we get the transcendental equations:

$$\rho_2 = \frac{1}{2\pi}\nu_1 + \frac{1}{\pi}\sum_{j=1}^{\infty} e^{-j^2\alpha\Delta t}\frac{\sin j\nu_1}{j}. \quad (18.2)$$

Solving this equation, for example, by the secant method, and using eq. (15.2), we get

$$\left.\begin{aligned} x_2 &= \cos\left[2\pi f_N(2 - N/2)\Delta t + \vartheta_2\right] \\ \vartheta_2 &= \vartheta_1 + \nu_1 . \end{aligned}\right\} \quad (19.2)$$

Similarly, we compute the phases $\vartheta_3, \vartheta_4, \ldots, \vartheta_N$ and the values of the parameters $x_3, x_4, \ldots x_N$ at the other points.

## § 3. Derivation of the Basic Computation Formulae

We shall now consider the general stationary process $X(t)$, which is ergodic relative to the first moment and correlation function and has a value of $x(t)$.

According to equations (35.1), (40.1) and (9.1), the simplest asymptotically indisplaced and consistent estimate

of the spectral density $W_x(f)$ at $T \to \infty$ is given by the following expression when $\Delta f_x$ and T are correlated by the corresponding relationship:

$$\tilde{\rho}_{x_T}(f) = \frac{2}{T \Delta f_x} \int_{f-\frac{\Delta f_x}{2}}^{f+\frac{\Delta f_x}{2}} \left\{ \int_{-\infty}^{+\infty} x_T(t) e^{i 2\pi \xi t} dt \right. \left. \int_{-\infty}^{\infty} x_T(t') e^{-i 2\pi \xi t'} dt' \right\} d\xi \qquad (1.3)$$

Introducing the new variable $\tau = t' - t$, and changing the order of integration inside the brace brackets, and also taking into consideration the even parity of the function $r_{x_T}(\tau)$ $= \frac{1}{T} \int_{-\infty}^{+\infty} x_T(t) x_T(t+\tau)\, dt$ and the definition (10.1) of $x_T(t)$, we get:

$$\tilde{\rho}_{x_T}(f) = \frac{4}{\Delta f_x} \int_{f-\frac{\Delta f_x}{2}}^{f+\frac{\Delta f_x}{2}} \left\{ \int_0^{T} \cos 2\pi \xi \tau \left[ \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cdot x(t+\tau)\, dt \right] \right. \left. d\tau \right\} d\xi \qquad (2.3)$$

Integrating over $\xi$, we finally get:

$$\tilde{\rho}_{x_T}(f) = \frac{4}{\Delta f_x} \int_0^{T} r_{x_T}(\tau) \frac{\cos 2\pi f \tau \cdot \sin \pi \Delta f_x \tau}{\pi \tau}\, d\tau, \qquad (3.3)$$

where

$$r_{x_T}(\tau) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}-|\tau|} x(t)\, x(t+|\tau|)\, dt \quad \text{for } 0 \le |\tau| \le T, \qquad (4.3)$$

$$r_{x_T}(\tau) = 0 \quad \text{when } |\tau| > T.$$

Since $x(t)$ is given in the finite range $-\frac{T}{2} \le t \le \frac{T}{2}$, then, the undisplaced and consistent estimate of the correlation function (22.1) can be rewritten as:

$$K_{x_T}(\tau) = \frac{1}{T-|\tau|} \int_{-\frac{T}{2}}^{\frac{T}{2}-|\tau|} x(t)\, x(t+|\tau|)\, dt \quad \text{for } 0 \le |\tau| \le T. \qquad (5.3)$$

From eq. (4.3) and (5.3) it is easy to see that

$$r_{x_T}(\tau) = K_{x_T}(\tau) \cdot \frac{1}{\left(1 - \frac{|\tau|}{T}\right)} \qquad (6.3)$$

Therefore, $\tau_{x_T}(\tau)$ is an asymptotocally undisplaced consistent estimate of the correlation function $K_x(\tau)$. The estimates $\tau_{x_{nT}}(\tau)$ and $r_{x_{nT}}(\tau)$ of the model process ($\alpha=0,6$ ; $f_н=0,7$ ; $\Delta t=0,1$) are compared in fig. 3.

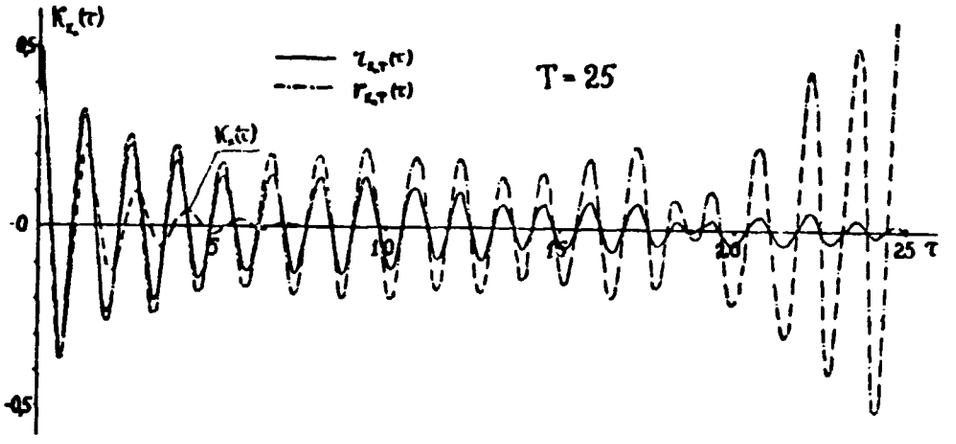



Fig. 3.

From the values of eq. (2.3) and (4.3), it is clear that

$$f_{xT}(f) = 4\int_0^T \cos 2\pi f\tau \cdot \tau_{xT}(\tau)\,d\tau = 4\int_0^\infty \cos 2\pi f\tau \cdot \tau_{xT}(\tau)\,d\tau, \qquad (7.3)$$

i.e., the periodogram $f_{xT}(f) \geq 0$ is correlated with $\tau_{xT}(\tau)$ by the Khinchin - Winer equation (see §1). It is noted that $\tau_{xT}(\tau)$ in eq. (7.3) is a consistent estimate while $f_{xT}(f)$ is not. The application of the Khinchin - Winer equation (13.1) and the consistent value of the correlation function does not, therefore, lead to a consistent estimate of the spectral density. This is due to (see 46.1) the decrease of the statistical reliability of the parameter $\tau_{xT}$ (the value of the distribution increases) on the increase of $\tau = m\,\Delta t$ at a fixed value of $T = N\,\Delta t$. It is easy to see that the expression of the consistent parameter $\varphi_{xT}$ (3.3) differs from (7.3) by introducing the weight factor $\frac{\sin \pi \Delta f_x \tau}{\pi \Delta f_x \tau}$ to $\tau_{xT}(\tau)$, which is closed to unity when is small, and which rapidly decreases when is increased. Thus, the weight of the so-called "tail" of the value if the correlation function in eq. (3.3) will decrease. If $r_{xT}$ is used in eq. (7.3), we will get undisplaced estimate of the spectral density $\varphi^H_{xT}(f)$ carrying the properties of the periodogram and having the possibility of acquiring negative values (see fig. 4 and 5).

However, this is not convenient because the estimated quantity $W_x(f)$ is essentially non-negative.

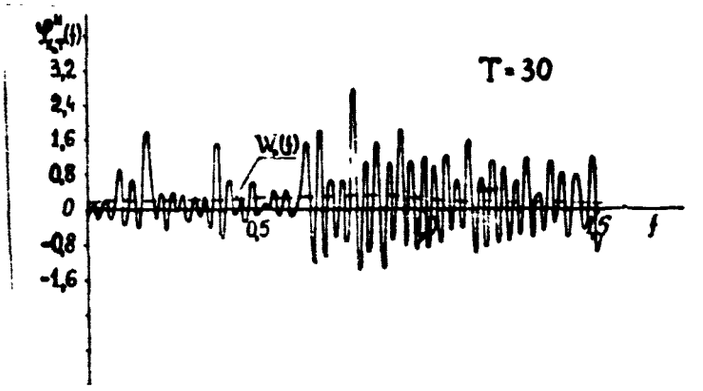



Fig. 4.

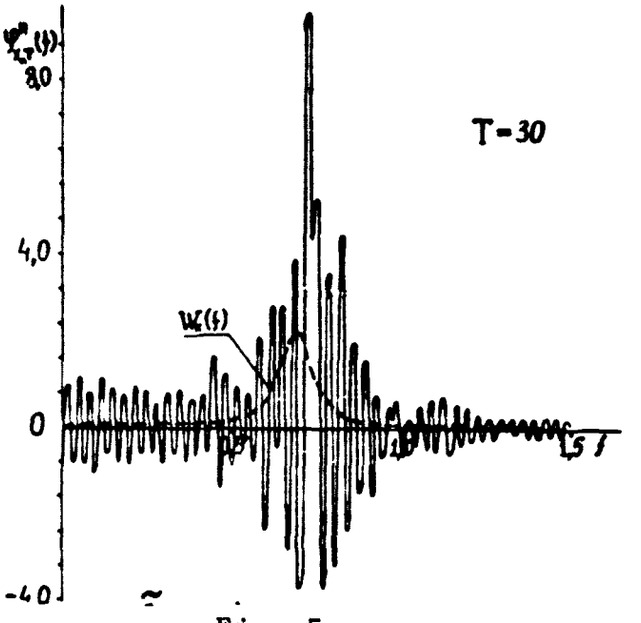



Fig. 5.

In computing $\bar{\Phi}_{xT}(f)$ by means of a computer, the main source of time loss is the time spent in the computation of the trigonometric functions. Thus, the use of eq.(3-3) instead of eq.(1-3) decrease the computation time by one or two orders of magnitude. This is because the computation of the integral over the periodogram

in the frequency range $\Delta f_x$ requires a large umber of points when $\Delta f_x T \gg 1$. In addition, eq. (3.3) is more suitable because it simultaneously leads to an estimate of the correlation function.

We take into consideration the ideal rectangular filter with a transmission band $(f - \frac{\Delta f}{2}, f + \frac{\Delta f}{2})$ and an asymptotically frequency character given by: $(f \geq 0)$

$$H(\psi) = \begin{cases} 1 \text{ at } f - \frac{\Delta f}{2} \leq |\psi| \leq f + \frac{\Delta f}{2}, \\ 0 \text{ at } |\psi| < f - \frac{\Delta f}{2} \text{ и } |\psi| > f + \frac{\Delta f}{2}. \end{cases} \quad (8.3)$$

Passing the signal $x(t) \left( \int_{-\infty}^{\infty} |x(t)| dt < \infty \right)$ with the amplitude spectrum:

$$S_x(\psi) = \int_{-\infty}^{\infty} x(t) e^{-i 2\pi \psi t} dt,$$

Through the filter, we get the transformed signal

$$[x(t)]^{\Delta f_x} = 2 \int_{-\infty}^{t} \frac{\cos 2\pi f (t-t') \sin \pi \Delta f_x (t-t')}{\pi (t-t')} x(t') dt' \quad (9.3)$$

having the amplitude spectrum:

$$[S_x(\psi)]^{\Delta f_x} = S_x(\psi) \cdot H(\psi).$$

Since $r_{xT}(\tau)$ is an even function, then by comparing eq. (3.3) with eq. (9.3), we get

$$\tilde{r}_{xT}(f) \cdot \Delta f_x = [r_{xT}(0)]^{\Delta f_x}. \quad (10.3)$$

Thus, the value of the correlation function at $\tau = 0$ in the frequency band $\Delta f_x$, which has an average frequency of f, is equal to the product of the spectral function and $\Delta f_x$.

We shall now consider the two general stationary processes X(t) and Y(t), which are ergodic relative to the first moment and the correlation function and which comprise the parameters x(t) and y(t). It will be assumed that the processes are rigidly connected and ergodic relative to the mutual correlation function (see § 1).

According to eq. (15.1), the quantity

$$\varphi_{xyT}(f) = \frac{2}{T} S_{xy}(f) \cdot S_{yT}^{*}(f) \qquad (11.3)$$

is an asymptotically undisplaced estimate of the mutual correlation functions $W_{xy}(f)$. However, it is inconsistent because the properties of its real and imaginary parts ($Re\,\varphi_{xyT}$ and $Im\,\varphi_{xyT}$) are analogous to the properties of the periodogram. This allows the application of the above mentioned procedure for the determination of $\tilde{\varphi}_{xyT}(f)$, which is the asymptotically undisplaced and consistent estimate of the mutual spectral density $W_{xy}(f)$. Like (1.3), when $T \to \infty$, we get:

$$\tilde{\varphi}_{xyT}(f) = \frac{2}{T \Delta f_{xy}} \int_{f-\frac{\Delta f_{xy}}{2}}^{f+\frac{\Delta f_{xy}}{2}} \left\{ \int_{-\infty}^{+\infty} y_T(t)\, e^{i 2\pi \xi t}\, dt \times \int_{-\infty}^{+\infty} x_T(t')\, e^{-i 2\pi \xi t'}\, dt' \right\} d\xi \qquad (12.3)$$

in the case when $\Delta f_{xy}$ is correlated with T by the corresponding relationships. Introducing the new variable $\tau = t' - t$, and taking into consideration that the function $r_{xyT}(\tau) = \frac{1}{T} \int_{-\infty}^{+\infty} x_T(t+\tau)\, y_T(t)\, dt$ is uneven, we get, in analogy with (2.3), the following estimate of the real part $R_e\, W_{xy}(f)$ of the mutual spectral density

$$Re\left[\tilde{\varphi}_{xyT}(f)\right] = \frac{2}{\Delta f_{xy}} \int_{-T}^{T} r_{xyT}(\tau) \cdot \frac{\cos 2\pi f \tau \cdot \sin \pi \Delta f_{xy} \tau}{\pi \tau}\, d\tau, \qquad (13a.3)$$

where

$$r_{xyT}(\tau) = \begin{cases} \frac{1}{T} \int_{\frac{T}{2}-\tau}^{\frac{T}{2}} x(t+\tau) y(t)\, dt & \text{when } 0 \le \tau \le T, \\ \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}-\tau} x(t+\tau) y(t)\, dt & \text{when } -T \le \tau \le 0, \\ 0 & \text{when } |\tau| > T. \end{cases} \qquad (14.3)$$

For the processes under consideration, the parameter $\tau_{xy\tau}(\tau)$ gives an asymptotically undisplaced and consistent value of the mutual correlation function $K_{xy}(\tau)$.

Since in (13a.3) the multiplication factor $\tau_{xy\tau}$ consists of an even function of $\tau$, then, by introducing the value of the intermediate mutual correlation function $K^{c}_{xy}(\tau)$. (see 18.1)

$$\tau^{c}_{xyp}(\tau) = \frac{\tau_{xyp}(\tau) + \tau_{xyp}(-\tau)}{2}, \qquad (15.3)$$

we can represent this relationship in the following form

$$R_e[\tilde{P}_{xyp}(f)] = \frac{4}{\Delta f_{xy}} \int_0^{\infty} \tau^{c}_{xyp}(\tau) \frac{\cos 2\pi f\tau \cdot \sin \pi \Delta f_{xy} \tau}{\tau} d\tau \qquad (15b.3)$$

The adaptation of this formula shortens the computation effort twice.

Using eq. (9.3), we get from (13.3) the following relationship

$$R_e[\tilde{P}_{xyp}(f)]\Delta f_{xy} = [\tau_{xyp}(0)]\Delta f_{xy} \qquad (16.3)$$

which is analogous to eq. (10.3). Thus, the value of the mutual correlation function at $\tau = 0$, in the frequency band $\Delta f_{xy}$ about the mean frequency f, is given by the product of the estimate of the real part of the mutual spectral density and

The following is a summary of the main results of this section: Equations (3.3), (4.3), (13.3) and (14.3) give a very convenient mathematical method for the determination (by means of a computer) of the asymptotically undisplaced and consistent estimate of the following statistical characteristics of the processes X (t) and Y(t):

$K_x(\tau)$ and $W_x(f)$,
$K_y(\tau)$ and $W_y(f)$,
$K_{xy}(\tau)$ and $W_{xy}(f)$.

Applying to these relations any suitable quadrature formula for

the computation of the integrals, we get the final expression for setting up the computer program. It is noted that the use of quadratures in eq. (3.3) and (13.3), which separate the trigonometric functions, is useful because $\zeta_{xT}(\tau)$ and $\zeta_{xyT}(\tau)$ have also usually a vibrating nature. In choosing the discretization step $\Delta t$ (see the next section), it is necessary to take into consideration the required accuracy of calculating the integrals.

## § 4. Some General Aspects of the Practical Application of the Method

In carrying out the computer calculations, it is necessary that the parameter $x(t)$ of the process under investigation is given at discrete points with a certain step $\Delta t$ . We thus convert the study of the random process $X(t)$ into a study of the discrete sequence $X_n = X(n\,\Delta t)$. The spectral properties of $X_n$ and $X(t)$ are, in general, different. In the numerical calculation, we are practically determining the statistical properties of the random sequence. Therefore, we have to select $\Delta t$ in such a manner that the error in the spectral function is reduced as much as possible.

We first ignore the effect of the finite period of the realization time and find out the relationship between the spectral density of the sequence $W_{x_n}(f)$ and that of the studied process $W_x(f)$. The spectral density $W_{x_n}(f)$ is defined by eq. (42.1) and is a periodic function with a period $\frac{1}{\Delta t}$ . The correlation function of the series (41.1) is a sequence of the values of the correlation function of the studied process at the discretization points $m\,\Delta t$ and is related to $W_{x_n}(f)$ by the Khinchin - Winer equations which have the form:

$$\left.\begin{aligned} K_x(m\Delta t) &= \frac{1}{2}\int_{-\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} W_{x_n}(f)\, e^{i2\pi f m\Delta t}\, df, \\ W_{x_n}(f) &= 2\Delta t \sum_{-m}^{+m} K_x(m\Delta t)\, e^{-i2\pi f m\Delta t} \end{aligned}\right\} \qquad (1.4)$$

Substituting $K_x(m\,\Delta t)$ into the last equation and changing the

order of summation and integration, we get:

$$W_{x_n}(f) = \Delta t \int_{-\infty}^{\infty} \left[ W_x(\xi) \sum_{m=-\infty}^{+\infty} e^{i 2\pi (\xi - f) m \Delta t} \right] d\xi .$$

Using the well-known equation

$$\sum_{m=-\infty}^{+\infty} e^{i 2\pi (\xi - f) m \Delta t} = \frac{1}{\Delta t} \sum_{k=-\infty}^{+\infty} \delta\left(f - \xi + \frac{k}{\Delta t}\right),$$

where $\delta(\xi)$ is the delta function, and changing once more the order of integration and differentiation, we finally get:

$$W_{x_n}(f) = \sum_{k=-\infty}^{+\infty} W_x\left(f + \frac{k}{\Delta t}\right). \qquad (2a.4)$$

A necessary condition for the uniform convergence of this series is the integrability of $W_x(f)$ (which is because $W_x \geq 0$) and the limitation of its variation on the whole infinite range $0 \leq f \leq \infty$.

Consequently, when we shift from the initial process to a random sequence, we should adjust the values of $W_x(f)$ for each value of f at points further than f from the right and left by an integer number of periods for the function $W_{xn}$. On the other hand, if the initial process X(t) has a spectrum of constant frequency, i.e. if $f \geq f_* > 0$, then $W_x(f)=0$, and it follows from the condition

$$\Delta t \leq \frac{1}{2 f_*} \qquad (3.4)$$

that there will be no correction for $W_x\left(f + \frac{k}{\Delta t}\right)=0$ or any $k \neq 0$ in this range, so that $W_{xn}(f) \equiv W_x(f)$. Taking into consideration the uniformity of the function $W_x(f)$, eq. (2a.4) can be rewritten as:

$$W_{x_n}(f) = W_x(f) + \sum_{k=1}^{\infty} \left[ W_x\left(\frac{k}{\Delta t} + f\right) + W_x\left(\frac{k}{\Delta t} - f\right) \right]. \qquad (2b.4)$$

We shall now consider the case when the spectrum of the studied random process X(t) does not have a constant frequency, and introduce the error $\Delta W_x = W_x - W_{x_n}$ of the spectral density defined by eq. (2b.4). We are usually interested in the spectral density in a certain frequency range; $0 \leq f \leq \Omega$. Let $W_{min}$ be the minimum value of $W_x(f)$ in this range. If the spectral density decreases by increasing the frequency quite rapidly so that, for $f \geq f_* > \Omega$, we can find the value of the non-increasing function $D(f) \geq W_x(f)$ whose series (2.4) converges,

then we shall get $\Delta W \leq W_{min}$ for quite large values of $f_*$ , i.e. a small violation of the spectral density takes place in the frequency range which are interested in. The error will be small when $W_x(f)$ decreases faster than $f \to \infty$ . In practice, the spectral density satisfies the required properties since $W_x(f) \geq 0$ and the real processes have a finite value, i.e. $J = \int_0^\infty W_x(f)\,df < \infty$. In addition, the high-frequency part of the spectrum is usually out off by the recording or transforming apparatus. It is thus useful if possible to transmit the studied parameter of the process through a wide-band filter in order to remove the high-frequency part of the spectrum. This allows to use a maximum discretization step.

The above arguments are mainly useful in studying wide-band processes. If we have a narrow-band process (see 8.2), i.e. the spectral density has a sharp peak in the narrow frequency range $f_0 - \frac{\Delta f_0}{2} < f < f_0 + \frac{\Delta f_0}{2}\ (\Delta f_0 \ll f_0)$ and has a small value outside this range, that tends to zero as $f \to \infty$ , then it will be recommended to select the discretization step $\Delta t$ in a different way. From e . (2.4) it is easy to see that in this case, the error caused by the transition from $W_x(f)$ to $W_{xn}(f)$ is small in the interesting range $\left( f_0 - \frac{\Delta f_0}{2} , f_0 + \frac{\Delta f_0}{2} \right)$ when the following condition is fulfilled at the arbitrary integral $K \geq 0$:

$$\frac{(K+1)/f_0}{2+\Delta f_0/f_0} > \Delta t > \frac{K/f_0}{2-\Delta f_0/f_0} \qquad (4a.4)$$

From eq. (4a.4), it follows that

$$K < \frac{2-\Delta f_0/f_0}{2\cdot\Delta f_0/f_0} . \qquad (4b.4)$$

Let $K_{max}$ be the maximum integer that satisfies eq. (4b.4). According to eq. (4a.4), it corresponds to a certain interval of possible values of $\Delta t$ ($K=0$ corresponds to condition 3.4). Increasing the narrow-bandedness degree of the process $\frac{\Delta f_0}{f_0}$ leads to the increase of $K_{max}$ and the decrease of the corresponding interval of the possible values of $\Delta t$. Thus, the use of condition (4.4) instead of (3.4) allows the increase of the discretization step of the narrow-band processes.

From the general theory (see § 1) and the analysis to be

done below in this section, it follows that the required realization time is only determined by the properties of the curve of $W_x(f)$ (the ratio of the caharacteristic values of $W_x$ and $W_x''$) in the frequency range under consideration, and not by the position of the peak. This allows to study very high narrow band processes with $\frac{f_0}{\Delta f_0} \gg 1$ without much loss of the compuier memory on the selection of $\Delta t$ by means of eq. (4.4).

As an example, we shall consider the case when $f_0 = 10^4 H_z$, $\Delta f_0 = 50 H_z$ and the form of the peak is such that $\Delta f_x \sim 5 H_z$ (see below). According to eq. (4.4), we may take $997.506 \times 10^{-5} > \Delta t_{max} > 997.494 \times 10^{-5}$. Since the distribution of the parameter $\Delta_2^2$ must be small (see eq. 52.1), then assuming that $W_x(f_0) \simeq O(1)$ we shall take $T \Delta f_x = 10$. Then $T_{min} = 2$ sec. and $N_{min} = \frac{T}{\Delta t_{max}} \approx 200$. On the other hand, if equation (3.4) or (4a.4) are used with K=0, we will get $(f_* = f_0 + \frac{\Delta f_0}{2})$ $4.987 \times 10^{-5} > \Delta t > 0$ and $N \approx 40105$. As mentioned above, in our tendency to use the maximum discretization step, we have to use an accurate value of $\Delta t_{max}$. In the example under consideration, when t=0.01 sec., the peaks will be superimposed and the error will be great. This means that it is necessary to specify the frequency range $(f_0 - \frac{\Delta f_0}{2}, f_0 + \frac{\Delta f_0}{2})$ such that the spectral density $W_x(f)$ is small outside it. If this range is not known beforehead, a preliminary computation should be carried out with the highest value of $\Delta f_0$ that the computer's memory allows.

It should be noted that the discretization process itself includes an element of randomness and is characterized by certain statistical properties. The influence of this factor on the accuracy of determination of the statistical properties of the studied process will not be considered here because this question is closely related to the concrete method of discretization and must be discussed in the course of application of this method; it could be considered that this influence is usually small.

The correct selection of the values of the realization time T and the average range of frequency $\Delta f_x$ in the equation used for estimation of the spectral density (3.3) is very important in order to obtain reliable results (see §1). In the initial solution of this problem, it is necessary to stipulate some important assumptions about the behaviour of the spectral density of this process. After the determination of the value of $W_x(f)$ and its main features are clarified, the values of T and $\Delta f_x$ can be selected more accurately, and the spectral density can be then recalculated.

Fig. 6 shows the nature of the dependence of the spectral

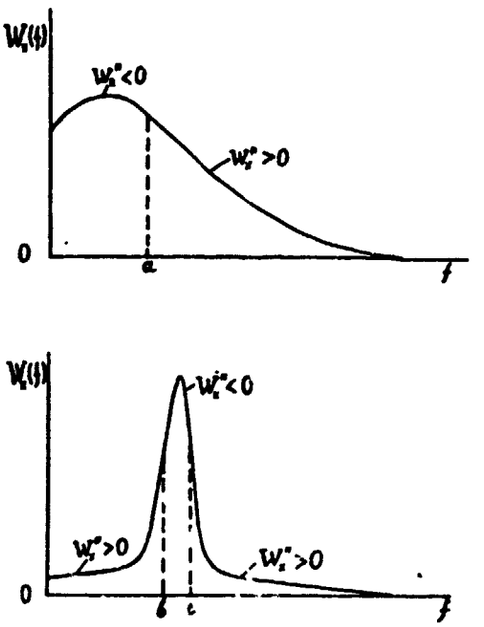



Fig. 6.

density on the frequency, ... wide- and narrow- band processes. Let the frequency range under consideration be divided by means of the points a,b and c into sections, where $W_x'' > 0$ or $W_x'' < 0$. The process is initiated by considering the section where $W_x > 0$. The maximum and minimum values of the spectral density in the considered section are denoted by $W_{max}$ and $W_{min}$, respectively, and the frequency scale $\alpha$ is selected from the condition:

$$W_x''(f) \leq \frac{2}{\alpha^2} W_{max}.$$

Then, by using eq. (51.1) for the determination of the displacement, we get:

$$\Delta_y \sim \frac{|W_{x_0}''|\alpha^2}{W_{max}} \cdot \frac{W_{max}}{24} \left(\frac{\Delta f_x}{\alpha}\right)^2, \qquad (5.4)$$

where

$$\frac{|W_{x_0}''|\alpha^2}{W_{max}} \leq 1$$

Using this equation, we choose the following average range of frequency

$$\Delta f_x < \alpha \sqrt{\frac{24}{W_{max}}}, \qquad (6.4)$$

which leads to a very small displacement. Defining the permissible distribution parameter $\Delta_2^2$ , we can use the following relationship (see 52.1)

$$\left.\begin{aligned} \Delta_2 \sim \left(\frac{W_{x_n}}{W_{max}}\right) \cdot \frac{W_{max}}{\sqrt{N \cdot \Delta t}\, \Delta f_x} \\ \frac{W_{x_n}}{W_{max}} \leq 1 \end{aligned}\right\} \qquad (7.4)$$

to determine the necessary value of N, i.e. the value of the realization ti e T=N $\Delta$t:

On th› variation of the value of the realization time, it is necessary to keep the parameters T and $\Delta f_x$ related by the condition $\frac{|\Delta_1|}{\Delta_2} = Const.$ . We have seen in §1 that the sum $|\Delta_1| + \Delta_2$ has a minimum value when $\frac{|\Delta_1|}{\Delta_2} = \frac{1}{4}$ since $W''_{x_n}$ decreases on the decrease of $W_{x_n}$ in the section where $W_{x_n} > 0$, so that the ratio $W_{x_n} / W''_{x_n}$ changes quite slowly, then, according to eq. (53.1), we get:

$$(N \Delta t)^{1/5} \Delta f_x = C_x$$

$$C_x = \left[6 \frac{W_{x_n}}{|W''_{x_n}|}\right]^{\frac{1}{5}} = \left(\frac{6}{B}\right)^{\frac{1}{5}}, \qquad (8.4)$$

where $B = \frac{|W''_{x_n}|}{W_{x_n}}$ is replaced by its mean value in the given range, and $C_x$ is a constant paratmeter.

To evaluate the parameter $C_x$, the relationship between the special density and the freqüency is approximated by the function:

$$W = M \cdot e^{\pm B \cdot f}$$

where M and $B = \frac{1}{W} \left| \frac{d^2 W}{df^2} \right|$ are constants. Using the tw› parameters $(W_x)_1$ and $(W_x)_2$ of the spectral density at the c‹ responding freqüencies $f_1$ and $f_2$ , we get:

$$\left.\begin{aligned} W_x(f) \approx W = (W_x)_1 e^{\pm \sqrt{B}(f - f_1)}, \\ \pm B = \frac{\ln (W_x)_2 - \ln (W_x)_1}{f_2 - f_1}, \end{aligned}\right\} \qquad (9.4)$$

where the "+" sign is taken for the range where the spectral density increases and the "-" sign for that where it decreases. Usually, $W_{max}$ and $W_{min}$ are taken as the two fixed parameters of the spectral density. Substituting the expression of $B$ into eq. (8.4), we get:

$$C_x = \left\{ 6 \left[ \frac{f_2 - f_1}{\ln(W_x)_2 - \ln(W_x)_1} \right]^2 \right\}^{\frac{1}{3}} \qquad (10.4)$$

Thus, when the spectral density changes more slowly, i.e. when the process is characterized by more wide-bands, the constant $C_x$ takes larger values. For wide-band processes, the range in which $W_x'' > 0$ ($f > a$ in fig. 6) is the basic one, and the constant $C_x$ chosen according to the method given above, can be used in the whole frequency range.

We shall now consider the narrow-band process for which the spectral density has a sharp peak (see fig. 6). In this case, the abovemethod should be first applied to the "wings" of the peak, i.e. to the range $f < B$ and $f > c$. It is noted that in this case, the parameter $C_x$ takes values much lower than those obtained for wide-band processes. The chosen values of T and $\Delta f_x$ permit a satisfactory description of the wide-band part of the spectrum, the determination of the frequency range in which the peak of the spectral density is located ($Bc$ in fig. 6), and the initial evaluation of the nature of the peak profile. After this, it is necessary to carry out a special calculation for that narrow frequency range by determining the values of T and $\Delta f_x$ from eq. (8.4) and specifying the optimum conditions of description of the peak. For the sake of orientation, we shall consider the following two characteristic peak profiles for the spectral density (see fig. 7).

$$W_x(f) = P_x \exp\left\{ -\frac{4(f - f_0)^2}{\Delta f_0^2} \right\} \qquad (11.4)$$

and

$$W_x(f) = \begin{cases} P_x \sqrt{\frac{(\Delta f_0)^2}{4} - (f - f_0)^2} \\ \text{when} \\ f_0 - \frac{\Delta f_0}{2} < f < f_0 + \frac{\Delta f_0}{2} \\ 0 \\ \text{when} \\ f < f_0 - \frac{\Delta f_0}{2} \text{ и } f > f_0 + \frac{\Delta f_0}{2}. \end{cases} \qquad (12.4)$$

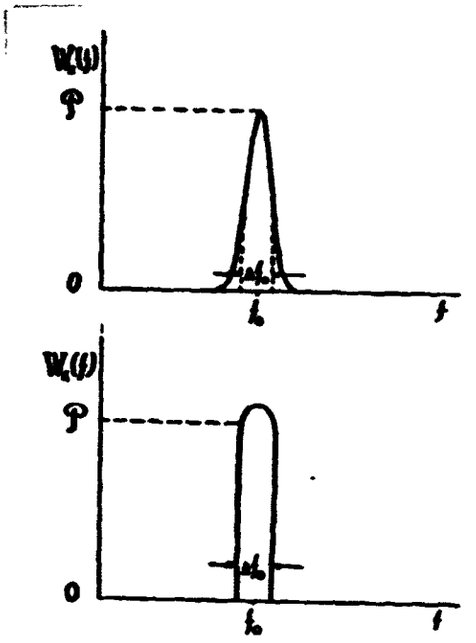



Fig. 7.

For the first (Gaussian) profile (11.4), $\Delta f_0$ is taken as a measure of the width, corresponding to the decrease of the spectral density in e-times. Evaluating $\beta = \frac{|W_x''(f_0)|}{W_x(f_0)}$, substituting into (8.4) and assuming that $\Delta f_x \sim \Delta f_0$, we get:

$$C_x = \left[\frac{9}{16}\Delta f_0^4\right]^{1/5}, \quad N\Delta t \sim \frac{1}{\Delta f_0} \quad \text{или} \quad N\Delta t \sim \frac{P}{J} \tag{13.4}$$

and

$$C_x = \left[9 f_0^2 \left(P + \frac{\Delta f_0}{2}\right)^2\right]^{1/3}, \quad N\Delta t \sim \frac{P^2}{\Delta f_0^3} \quad \text{или} \quad N\Delta t \sim \frac{P^5}{J^3} \tag{14.4}$$

(14.4),

where

$$J = \int_0^\infty W_x(f)\,df \tag{15.4}$$

is the power included in the peak. If the peak width $\Delta f_0$ increases while the maximum value remains constant, then the realization time increases, and the increase is more for the second profile. If the power $J$ remains invariable while

only the frequency scale of the process changes, then the dependence of the necessary relaization time on $\Delta f_0$ will remain the same for the first profile (the Gaussian) and will be considerably strengthened for the second one. Real profiles are probably more similar to the Gaussian profile in nature.

It could be thus seen that in order to get an estimate for the spectral density near the peak with a small displacement and distribution, we should use a realization time of a considerably large value. This is one of the most difficult points in getting reliable data about the maximum value of the spectral density of a narrow-band process.

What would be the case when the possible realization time is restricted by the computer's memory while the derivation of reliable estimates for the considered process requires a time of $T \gg T_0$? In this case we recommend to derive an estimate for a number of $\ell \left(\ell \gg \frac{T}{T_0}\right)$ successive sections of a time $T_0$ and then takes the average value of these $\ell$ estimates. This procedure is based on the fact that the relationship between neighbouring sections of the parameter decreases when its value increases. If the periodograms are calculated for each of the $\ell$ neighbouring sections of a time $T_0$, it could be then shown (Bartlett's equation) that when $T_0$ and $\ell$ increase, the average value of these periodograms will asymptotically give an undisplaced and constistent estimate for the spectral density.

By using the estimates (3.3), which are calculated for the sections $T_0$ instead of the periodograms, it will be possible during their averaging to utilize the data more completely and thus to take smaller values of $\ell$ and $T_0$.

To conclude the question about the choice of T and $\Delta f_x$, it should be pointed out that the proposed methods of choice are based on the asymptotic relations (47.1) and (48.1) which are derived for idealized random processes. In practical calculations, these parameters should be adjusted in order to get more accurate results.

## §5. The Results of Application of the Method to the Study of Model and Real Random Processes.

Using a model process (see § 2), we studied how the accuracy of determining the value of the spectral density $\varphi_{xT}(f)$ can be influenced by such factors as the discretization step $\Delta t$, the realization time T, the periodogram's average range of frequency $\Delta f_x$, the narrowness or wideness of the bands d of the studied process, the variation of the carrier frequency $f_H$ of the process, ... etc. In this section, examples are given to illustrate the results of these studies examples are also given for the analysis of the values of the real random processes obtained in the experiment .

The values of the realization times, the estimates of the correlation functions, as well as the periodograms and the estimates of the spectral densities were calculated at discrete points with steps of $\Delta t$, $\Delta\tau$ and $\delta f$, respectively. Then the uniform curves given in the figures were constructed from these points. The discretization step was taken equal to $\Delta t=0.1$ for all model processes except for the cases to be specially mentioned. For the values obtained in the experiment, $\Delta t=0.0004$ sec., and the step $\Delta\tau$ for calculation of the values of the correlation functions was taken equal to 0.1 sec. for model processes and 0.0004 sec. for real processes. The periodograms and the estimates of the spectral densities were calculated with a step $\delta f \leq \frac{1}{2T}$ in order to show the exact nature of the statistical fluctuations. In all the figures, the estimates $r_{x_nT}(\tau)$, $\varphi_{x_nT}(f)$, ... etc., are given by solid curves, whereas the unknown functions $K_{x_n}(\tau)$, $W_{x_n}(f)$, ... etc, are given by dashed curves. The curves give the realization time T and, if necessary, the average range of frequency $\Delta f_x$ and the parameter $C_x$.

Fig. 8 shows typical parts of realizations of wide-band ($d=5.5$ , $f_H=0.7$) and narrow-band ($d=0.45$ , $f_H=0.7$) model processes as well as wide-band and narrow-band real processes. For model processes, the ordinate $x(t)$, the time t and the frequency f are dimensionless.

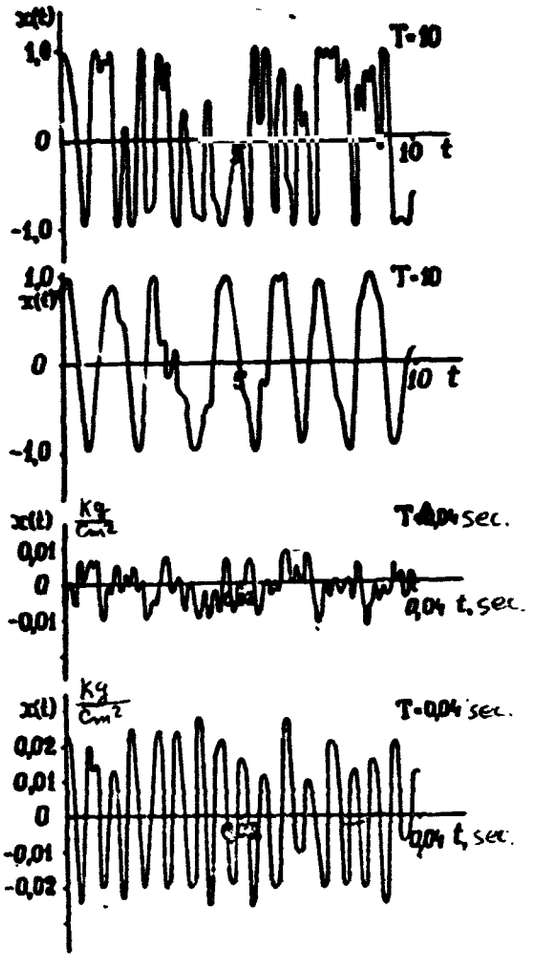



Fig. 8

Fig. 9 shows examples of periodograms for a wide-band model process, d=5.5 and $f_H$ = 0.7. The comparison with the unknown spectral density of the process, $W_x(f)$, (the dashed curves) shows the random nature of the periodogram (the solid curves).

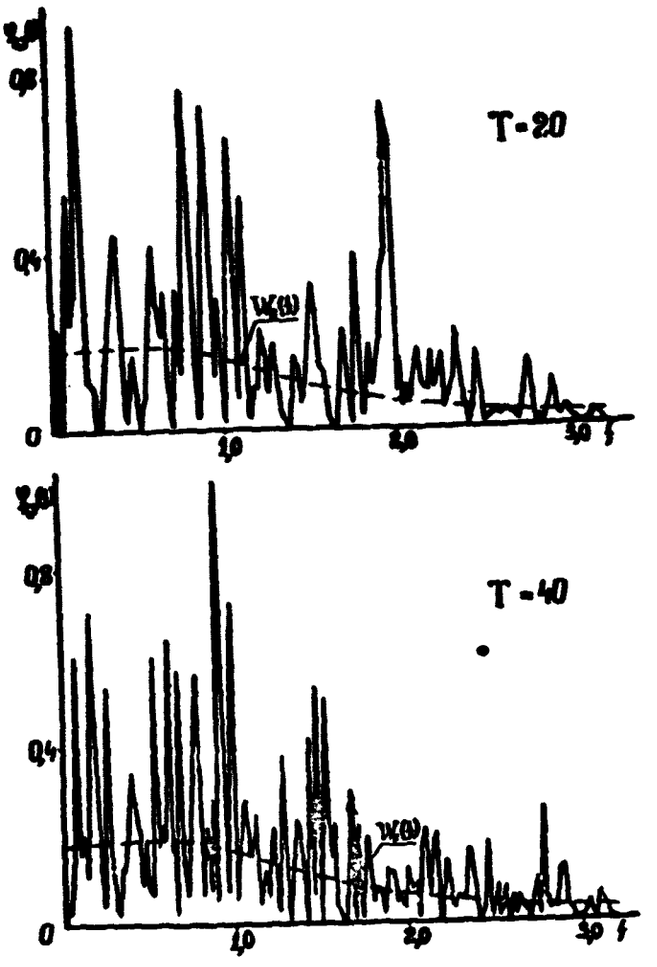



Fig. 9.

An important character of the random process is its correlation function, and its estimate is given by equations (4.3) and (5.3). Fig. 10 shows the convergence of the undisplaced parameter $r_{x_n T}(\tau)$ (solid curves) to the exact correlation function $K_x(\tau)$ (dotted curves) for the narrow-band model process ($\bar{d}$=0.45; $f_H$ =0.7). The parameter rapidly converges on a small initial section $\tau$, and the convergence deteriorates outside this section, where there is what is called "tail".

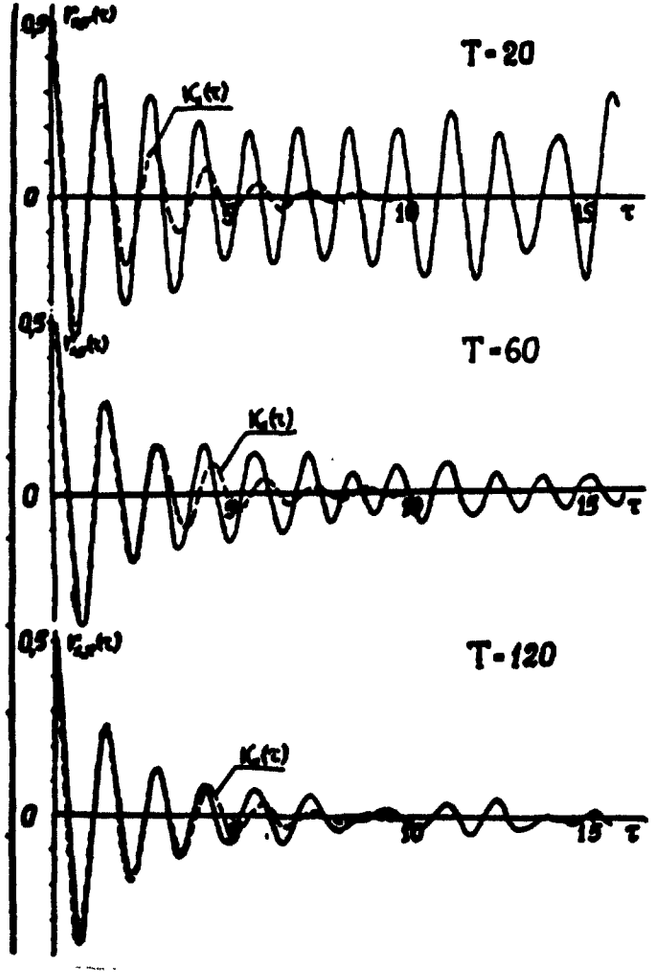



Fig. 10.

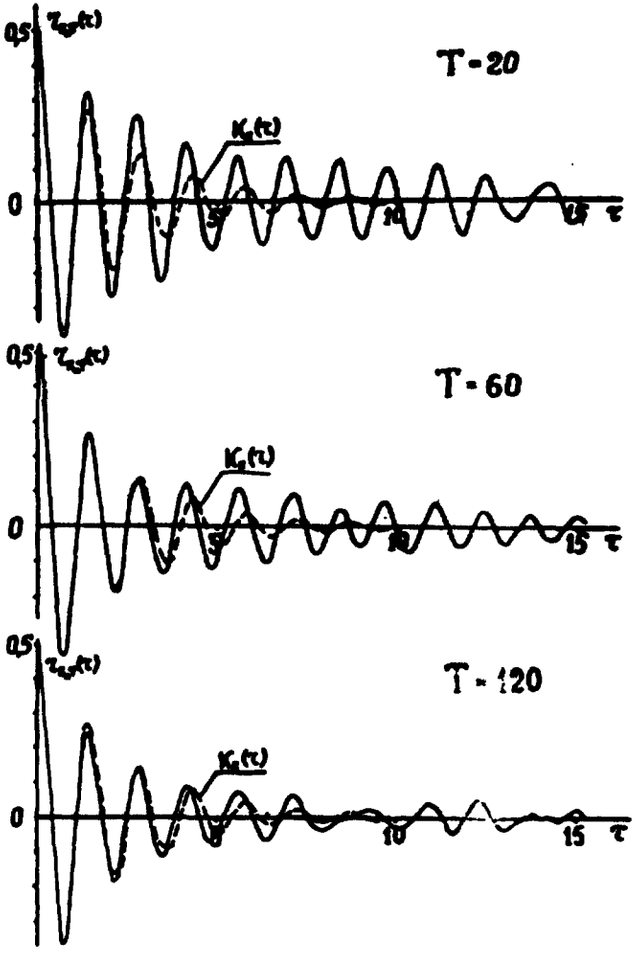
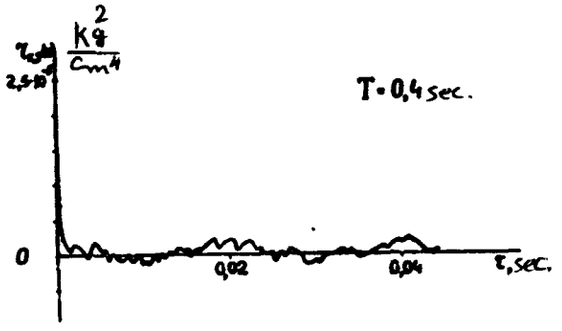



Fig. 11.

This behaviour is in agreement with equation (46.1) for the distribution of this parameter ($\tau = m\Delta t$). Analogous curves for the asymptotically undisplaced parameter $\tau_{x_nT}(\tau)$ are given in fig. 11. It is seen that the two parameters have a close convergence character due to their relation to each other (eq. 6.3). Fig. 3 gives a comparison between the parameters $r_{x_nT}(\tau)$ and $\tau_{x_nT}(\tau)$ for the other narrow-band process (with a wider frequency band d=0.6; $f_H$=0.7). It is noted that, for the narrow-band processes under consideration, the range of $\tau$ where the parameters are satisfactorily accurate is only 5-10% of T.

Fig. 4 and 5 show the values of the spectral density $\tilde{\varphi}^H_{x_nT}(f)$ obtained for wide- (d=5.5; $f_H$=0.7) and narrow- (d=0.45; $f_H$=0.7) band model processes, by substituting the undisplaced value of the correlation function into the Khinchin-Winer equation (eq. 13.1). From the figure it is seen that $\varphi^H_{x_nT}(f)$ has the characteristics of the periodogram and, in addition, takes negative values. It should be reminded that, in substituting $\tau_{x_nT}(\tau)$ into (13.1), we get the periodogram $\varphi_{x_nT}(f)$ (see (7.3) and the dicussion following it).



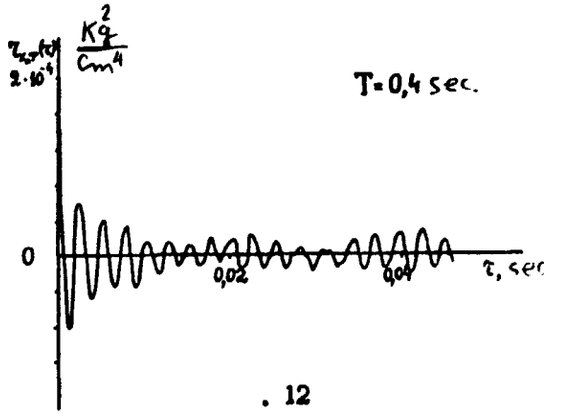




Fig. 12 gives the estimate $\tau_{x_nT}(\tau)$ of the correlation function for two real processes: a wide-band noise and the sum of a narrow-band process and the noise. As in the case of model processes, for a finite value of T, the estimate is valid only in a certain initial section of $\tau$ (till the place where the estimate's "tail" starts).

The estimate of the spectral density $\tilde{\varphi}_{x_n T}(f)$, which is determined by discretized realization, is a periodic function of the frequency (see §1) with aperiod $\frac{1}{\Delta t}$. Fig. 13 shows one period of an estimate calculated for a real narrow-band process. Here and hereafter, the vertical line separates half-periods of the estimate. According to the results of §4, the estimate corresponds to the unknown spectral density $W_x(f)$ in the range of $0 \leq f \leq f_в$ for the wide-band process. On choosing the discretization step $\Delta t$ according to eq. (3.4), and in the range of $f_0 - \frac{\Delta f_0}{2} < f < f_0 + \frac{\Delta f_0}{2}$ for the narrow-band process on choosing $\Delta t$ according to (4.4), i.e. in the range where $W_{x_n}(f)$ coincides with $W_x(f)$ within the required accuracy.

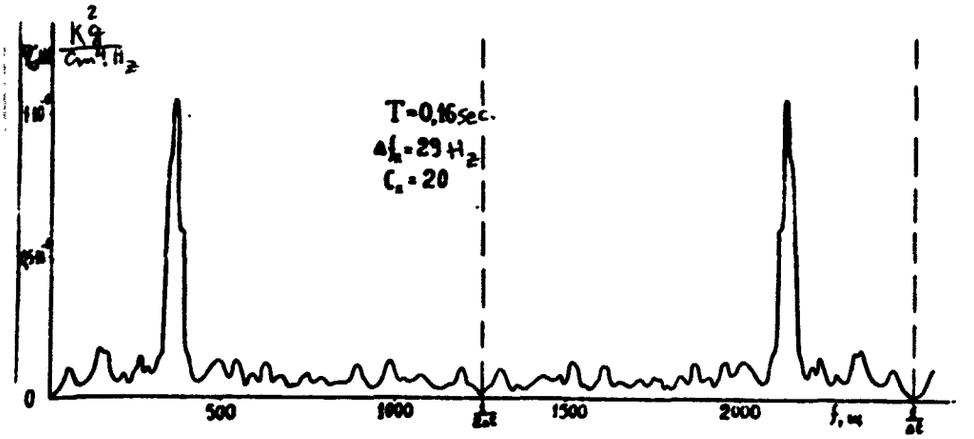



Fig. 13.

Fig. 14 shows the influence of the correct choice of the discretization step $\Delta t$ on the accuracy of the estimate of the spectral density for a wide-band model process ($d=5.5$; $f_H=0.7$). In the lower parts of the curves, the solid curves represent estimates with correct choice of $\Delta t$ (see §4), and the dashed curves show the unknown spectral density of the process $W_x(f)$.

The upper parts of the curves represent the parameter $W_{xn}(f)$ by dashed curves. The dashed-dotted curves are the corresponding estimates with incorrect (overestimated) values of $\Delta t=0.4$ and 0.5. The dashed curves with circles show the contributions of one, two, three and several terms of eq. (2.4).

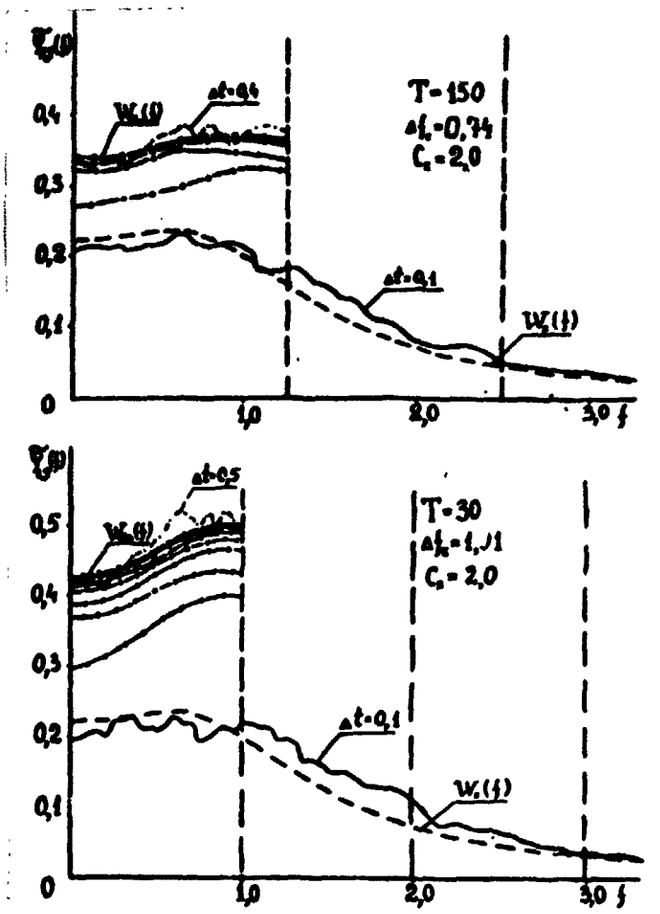



Fig. 14.

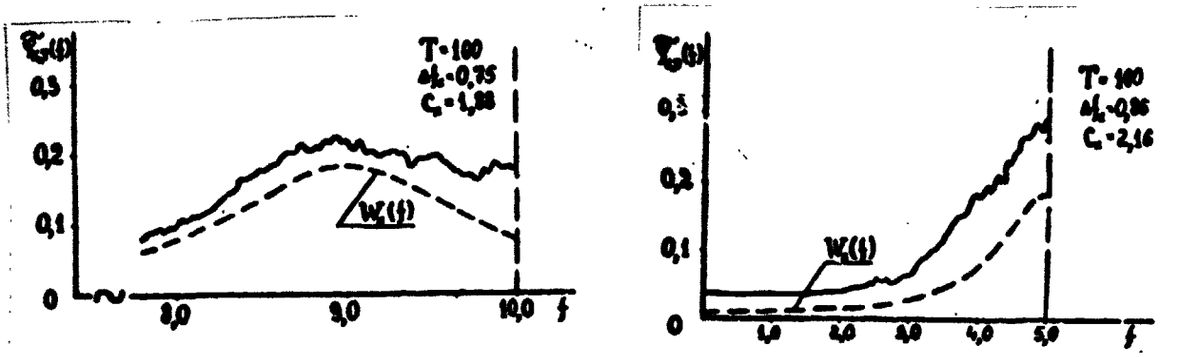



Fig. 15

Fig. 15 shows the effect of the position of the maximum value of the spectral density of two wide-band model processes (d=5.5, $f_H$=9.0 and $f_H$=5.0) on the value of the error caused by the incorrect choice of the discretization step $\Delta t$. By comparing the figures it is clear that the incorrect choice of $\Delta t$ may make the error quite significant.

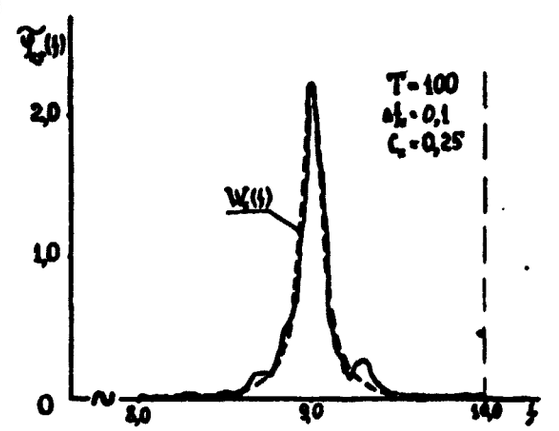



Fig. 16.

The method of selection of $\Delta t$ for a narrow-band model process (d=0.45, $f_H$=9.0) (see 4.4) is illustrated in fig. 16. It is easy to see (↑ §4) that eq. (4.4) is valid for the case shwon in fig. 16, and that the resulting estimate is a good approximation of the spectral density. Moreover, $\Delta t$ can be increased till $\approx$ 0.473 without violating the inequalities. If the discretization step is chosen by eq. (3.4), then it will be necessary that $\Delta t \leq 0.05$. Thus, the selection of $\Delta t$ by using eq. (4.4) instead of (3.4) for the narrow-band process allows in our case

to increase the discretization step by approximately 10 times. One should however take into consideration that the selection of $\Delta t$ by eq. (4.4) yields good results only in the frequency range ( $f_0 - \frac{\Delta f_x}{2}$ , $f_0 + \frac{\Delta f_x}{2}$ ). According to the results of §4, the estimate has spurious peaks outside this range (see 2.4).

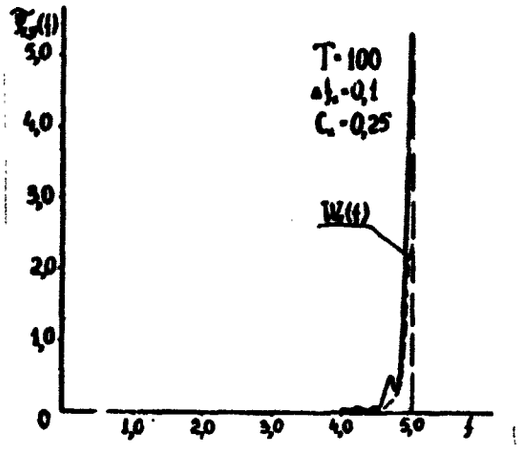



Fig. 17.

Fig. 17 shows an example of the incorrect choice of the discretization step (d=0.45, $f_н$=5.0). In this case($f_0$=5.0, $f_0 \approx 1.0$, t=0.1) even eq. (3.4) is not satisfied ($f_* \approx 6.0$). As a result, the error reaches 100%.

In order to get a reliable estimate for the spectral density $W_x(f)$, the selection of the realization time T=N $\Delta t$ and the average range of frequency $\Delta f_x$ play a decisive role (see § 4).

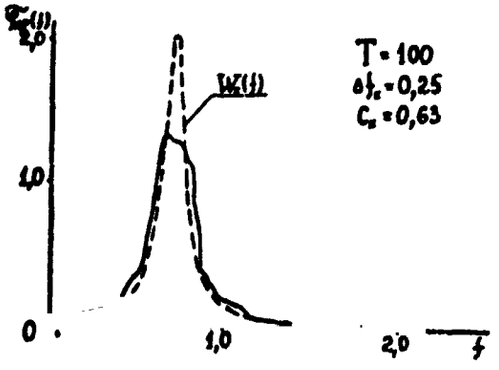



Fig. 18.

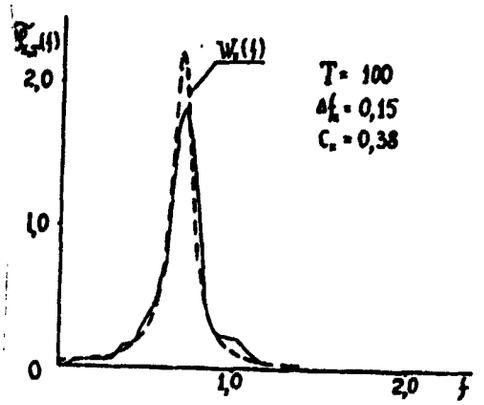



Fig. 19.

The methods used for the selection of T and $\Delta f_x$ and their influence on the accuracy of a narrow-band model process (d=0.45, $f_H$=0.7) are illustrated in fig. 18-25.

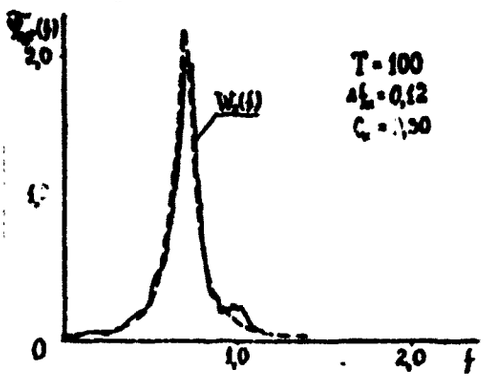



Fig. 20.

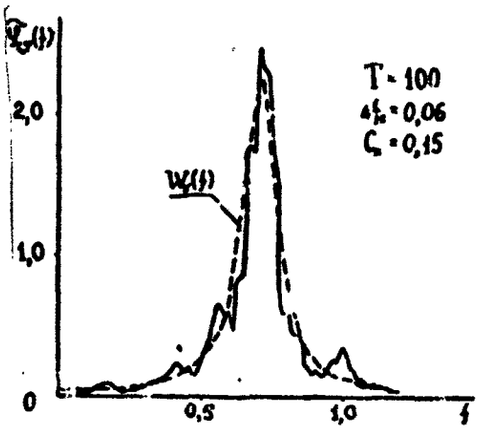



Fig. 21.

From figures 18,19,20 and 21 it is clear that, in accordance with the theory (see 5.4 and 7.4), the decrease of $\Delta f_x$ at a fixed value of T, and consequently the decrease of $C_x$, leads to a decrease and an increase in the values of the displacement and the distribution, respectively. The optimum value of $C_x$ lies between 0.30 and 0.15 (closer to 0.30).

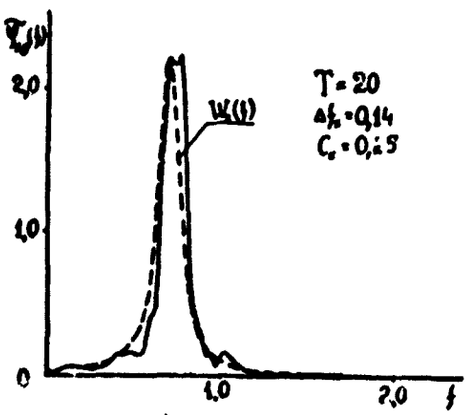



Fig. 22.

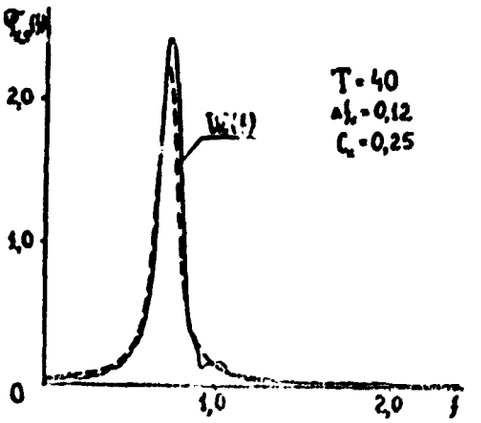



Fig. 23.

Figures 22,23 and 24 show that, at fixed (optimum) $C_x$, the increase of the realization time T (and consequently the decrease of $\Delta f_x$) leads to a decrease in the values of both the displacement and the distribution. In order to get a good estimate for the spectral density, a realization time of $T_{min} \approx 6000\ \Delta t = 60$ is necessary.

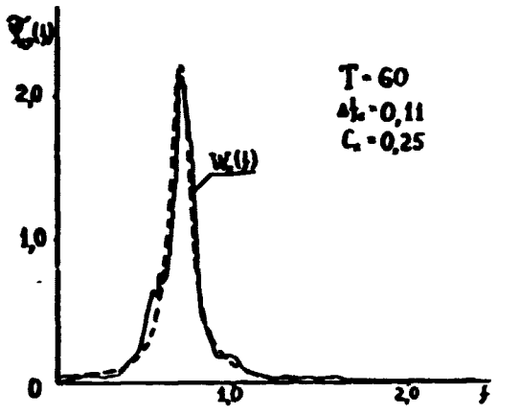



Fig. 24.

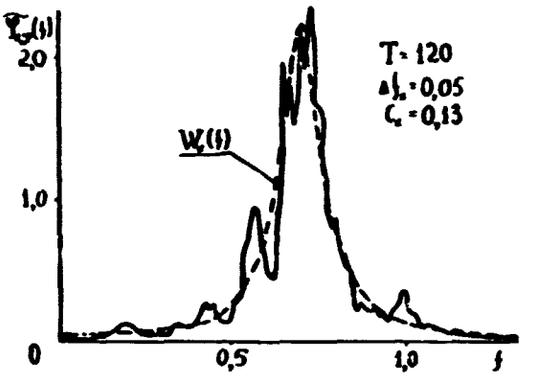



Fig. 25

Fig. 24 shows the estimate at $C_x = 0.25$, which agrees well with the unknown value. Fig. 18 and 25 show that it is impossible to get a good estimate for the spectral density with a considerable deviation from the optimum value of $\Delta f_x$ (and consequently $C_x$) even by using a very long realization time because of the occurrence of either a large displacement or a large distribution. Fig. 26 shows the calculations used for a model process whose frequency distribution of the spectral density has the same form but displaced towards lower frequencies (d=0.45; $f_H$=0.2). The calculation is done for an optimum value of $C_x$ equal to 0.255.

Fig. 27-31 show the results of analogous calculations for the case of a wide-band model process (d=5.5; $f_H$=0.7). The optimum value of $C_x$ is 1.82-1.88. Fig. 28-31 illustrate the growth of the distribution with the decrease of $C_x$ beyond the optimum value. The role of displacement is less in the case of wide-band processes because of the smooth variation of $W_x(f)$.

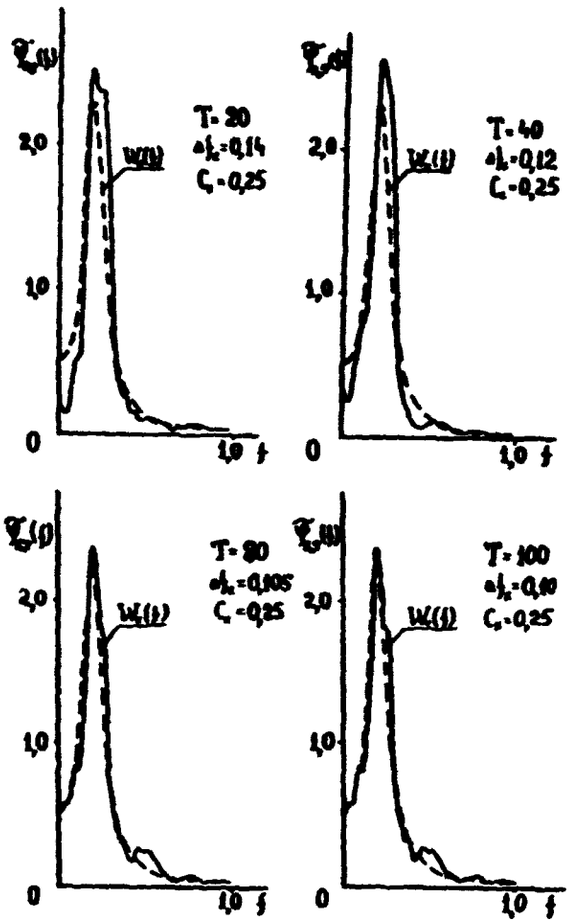

T=20
$\sigma_k$=0,14
$C_v$=0,25
T=40
$\sigma_k$=0,12
$C_v$=0,25
T=80
$\sigma_k$=0,105
$C_v$=0,25
T=100
$\sigma_k$=0,10
$C_v$=0,25
W(f)
2,0
1,0
0
1,0 f

Fig. 26.

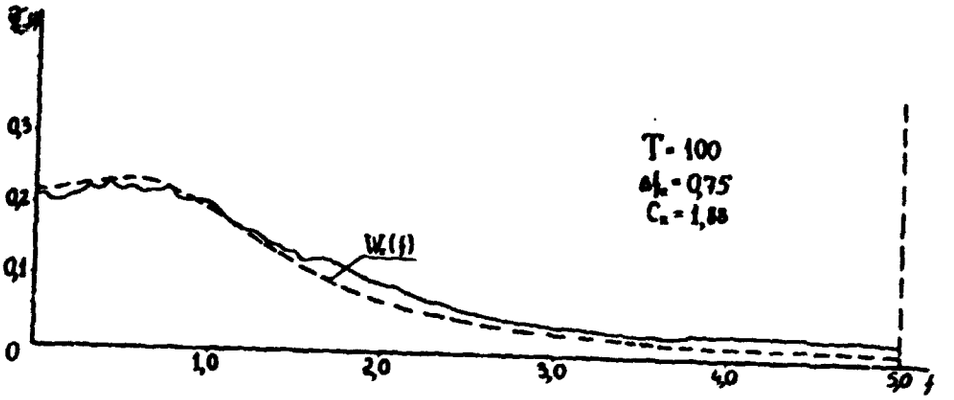

T=100
$\sigma_k$=0,75
$C_v$=1,88
W(f)
0,3
0,2
0,1
0
1,0
2,0
3,0
4,0
5,0 f

Fig. 27.

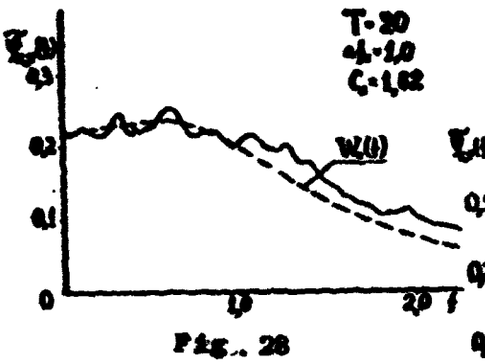



Fig. 28

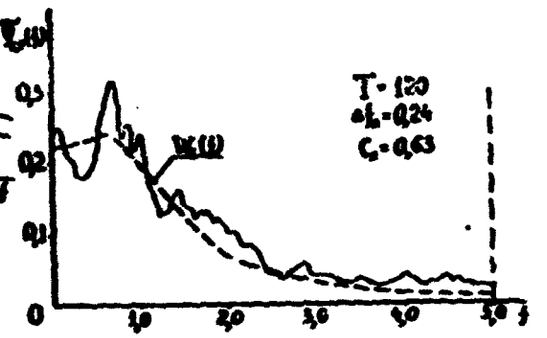



Fig.29

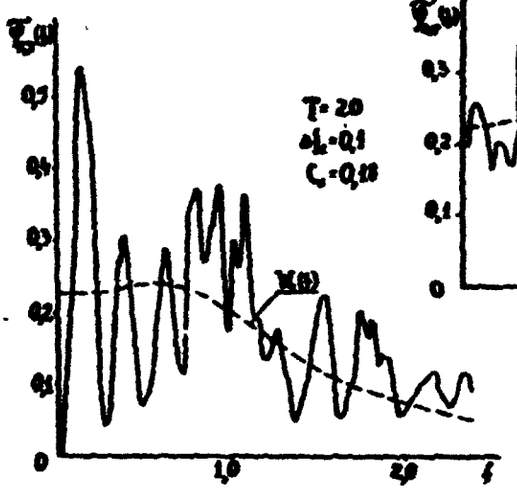



Fig. 31

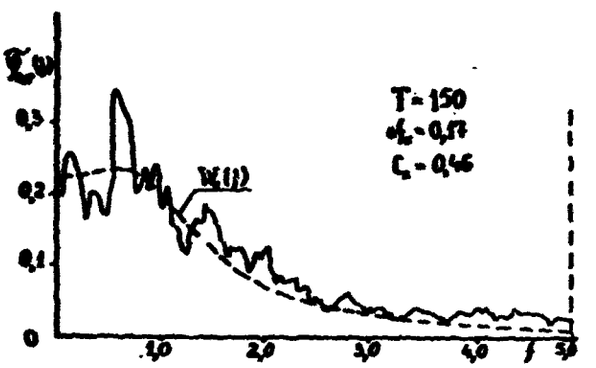



Fig .30

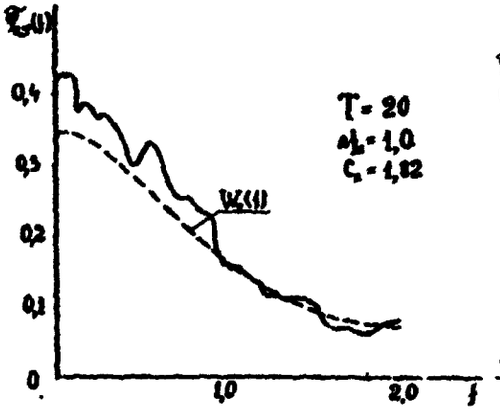



Fig. 32

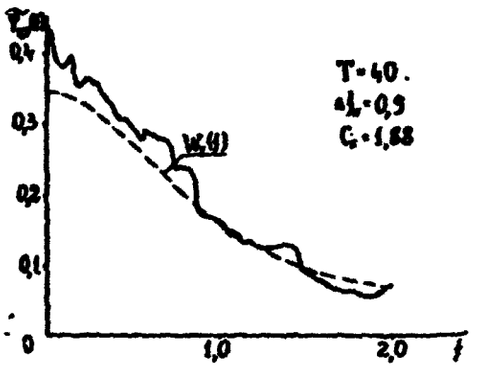



Fig. 33

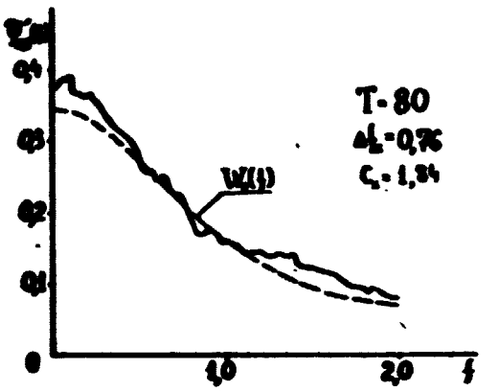



Fig. 34.

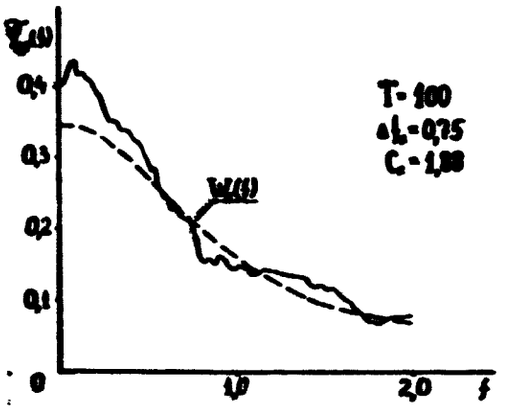



Fig. 35.

Fig. 32-35 show the application of the selected value of $C_x$ to the analysis of the model process whose frequency distribution of spectral density has the same form but displaced towards lower frequencies (d=5.5; $f_H$=0.2).

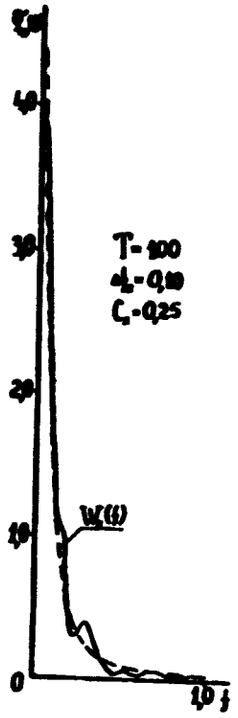



Fig. 36

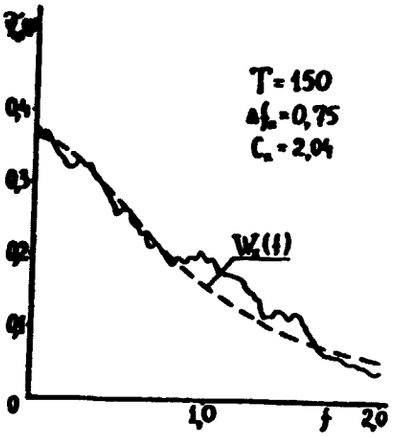



Fig. 37.

Fig. 3[illegible] and 37 give the values of the spectral densities of two wide-band processes with a very low carrier frequency $f_H$=0.007. At this frequency, the realization time under consideration has a space for only 0.7 vibration of the first process (d=0.45)(see fig. 36) and 1.05 vibration of the second one (d=5.5)(see fig. 37). Moreover, the data given in this section suggest that in both cases the parameter T can be reduced.

By comparing fig. 24 and 26 of the narrow-band processes and fig. 28, 31-37 of the wide-band ones, it is clear that the optimum value of $C_x$, as well as the corresponding values of the realization time T and the average range of frequency $\Delta f_x$, are independent of the value of the carrier frequency $f_H$, or on the location of the frequency range in which the main part of the spectral density is concentrated. In accordance with the general theory (see §1 and §4), the parameters $C_x$, T and $\Delta f_x$, and consequently (see 5.4 and 7.4), the values of the displacement $|\Delta_1|$ and the mean square deviation $\Delta_2$, will depend only on the properties of the curve (the characteristic values of $W_x$ and $W_x''$) in the frequency range under consideration.

By comparing the estimates given in fig. 28 and 32, and fig. 27 and 35, calculated for two processes with similar values of $W_x(f)$, we see that for equal values of $C_x$, T and $\Delta f_x$, and consequently, for close values of displacement and dispersion, the accuracies can be significantly different. The curves in fig. 32-35 clearly demonstrate that for a given realization time of the process, the increase of T can lead to a non-monotonous decrease in the estimate's accuracy. These two effects are due to the limitation of the estimate's distribution at a given realization time. In the case shown in fig. 32-35, the chosen realization time is less suitable for the corresponding values of T. We can only guarantee a small estimate error for very long realization times, when the dispersion is practically equal to zero for correctly chosen values of $\Delta f_x$.

As an example, the numerical method was applied for the analysis of some real processes obtained in the experiment. Some of the calculated data are given in figures 38-45. Fig. 38 gives the values of the spectral densities of the following two processes: the wide-band process X(t) and the narrow-band process Y(t). The mutual correlation value of these processes is given in fig. 39.

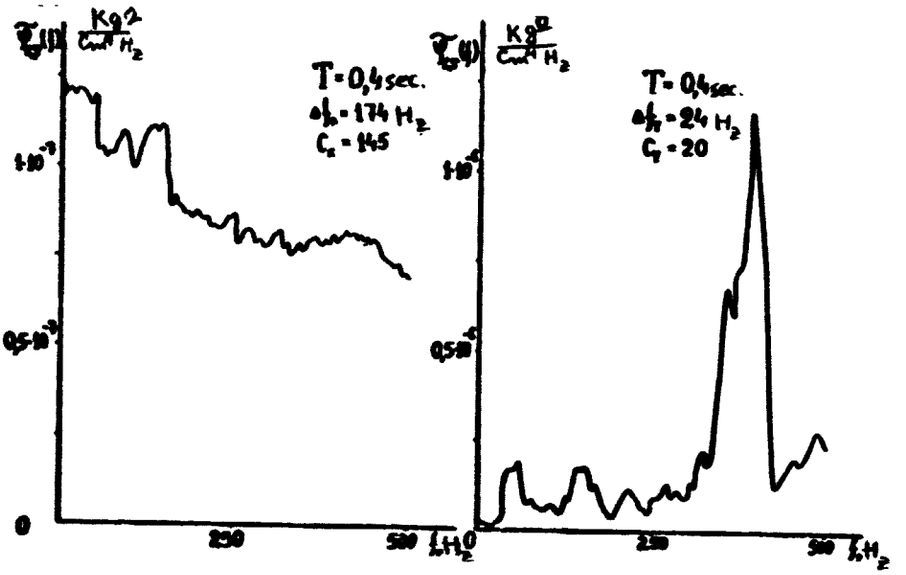



Fig. 38

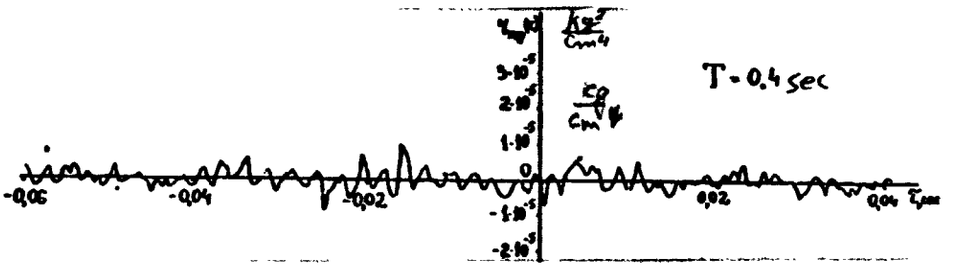



Fig. 39.

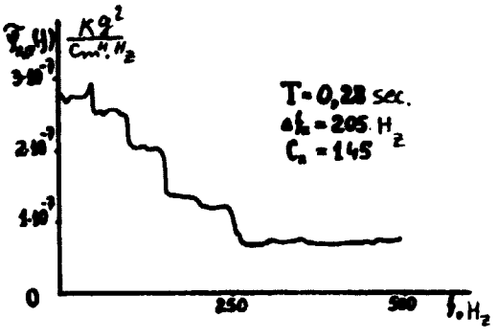



Fig. 40.

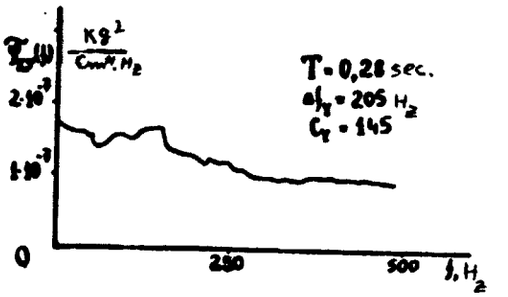



Fig. 41

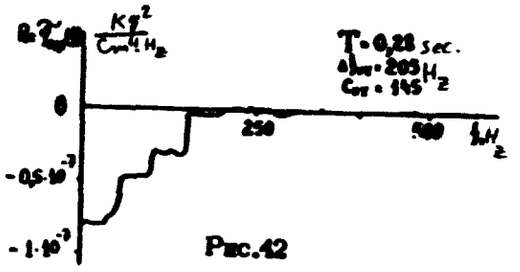



Рис.42

Fig. 42.

Fig. 40 and 41 show the values of the spectral densities of the two wide-band processes X(t) and Y(t), as well as the value of the real part of the mutual spectral density (see fig. 42). The same values, obtained for smaller values of the parameters $\Delta f_x$, $\Delta f_Y$ and $\Delta f_{xy}$, are given in fig. 43-45.

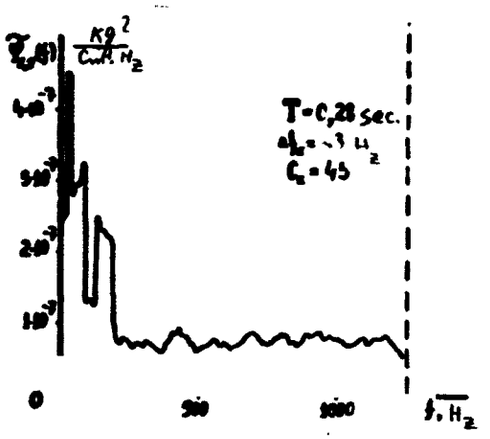



Fig. 43.

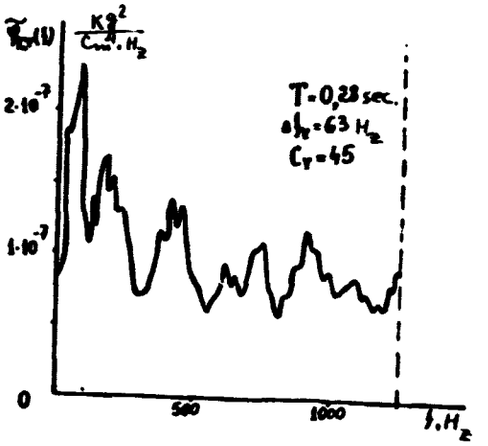



Fig. 44.

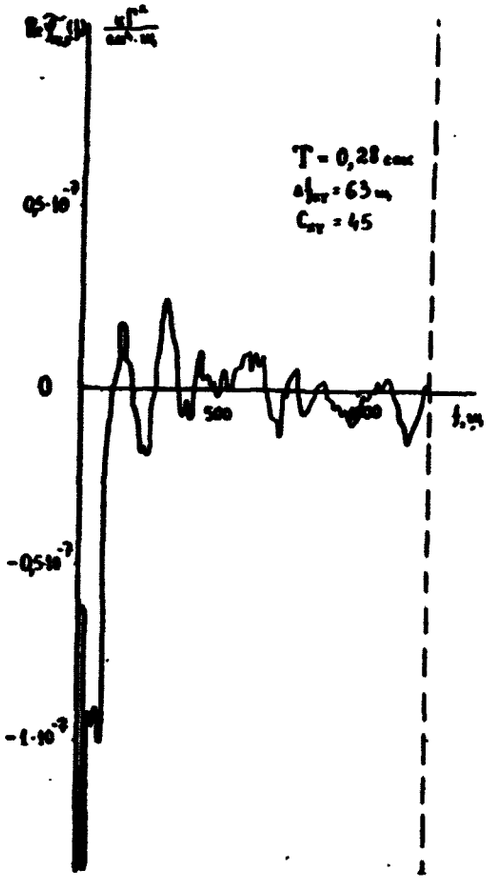



Fig. 45.

By comparing fig. 40 and 41 with fig. 42 and 44, it is clear that, in accordance with the general theory, the increase of $\Delta f_x$ and $\Delta f_y$ at a given realization time leads to an increase in the estimate's distribution. The decrease of $\Delta f_{xy}$ has an analogous influence (see fig. 42 and 45).

The calculation data reported in this section give an evidence of the efficiency of the numerical method. The problem of the optimum selection of the discretization step, the average range of frequency and the realization time by using a low-power computer (of an M-20 type) was also considered. By means of this computer it was possible, with a sufficient accuracy, to study the spectra of the random processes in both the low- and nigh- frequency bands. The computation time required for the calculation of the spectral density (3.3) at a given realization time can be evaluated by using the following equation:

$$T^{w}_{comp} \approx (2N^2 + 32Np) \cdot 10^{-6} [\text{min}],$$

where $N = \frac{T}{\Delta f}$, and p is the number of calculation points in the frequency range under consideration. It is noted that the computation time can be significantly reduced for the same accuracy degree by using the value of the special weight function $B_T(\xi)$ given in eq. (37.1) instead of the given in eq. (39.1).

## CONCLUSIONS

1. Using the general theory of the statistical estimates of the characteristics of random stationary processes, a numerical method convenient for computer use has been selected for the determination of the simple and mutual spectral densities and the correlation functions from the parameters of the studied processes. A computer program has been constructed and a number of random processes have been analysed.

2. By the numerical analysis of wide- and narrow-band model stationary processes by means of a computer, the condition of applicability of the method and the basic factors influencing its accuracy have been studied in detail. The problems of the optimum choice of the values of the discretization step, the average range of frequency and the realization time were discussed.

3. The obtained results indicate the efficiency of the numerical method. A low-power computer (of an M-20 type) could be used for the study of the spectra of random processes in both the low- and high-frequency bands with a sufficient accuracy.

4. Recommendations concerning the basic questions raised in this chapter are given. Thus, the method could be regarded as a short manual for the practical application of numerical methods for the spectral and correlative analyses of random processes by means of computers.

## References

1. Bartlett M.S., "Vvdeniye v teoriyu sluchaynykh protsessov". (Introduction to the theory of random processes). - - - Izd. inostr. li:., Moskva, 1958.

2. Yaglom A.M., Vvedeniye v teoriyu statsionarnykh sluchaynykh funktsiy" (Introduction to the theory of stationary random functions. --- (UMN) 7-5 (51), 1952.

3. Pugachev V.S., "Teoriya sluchaynykh funktsiy" (Theory of random functions). - - - Fizmatgiz, Moskva, 1962.

4. Middlton D., "Vvedeniye v statisticheskuyu teoriyu svyazi" (Introduction to the statistical theory of communication). - - - Izd. "Sovetskoye Radio", vol. 1, 1961, and vol. 2, 1962.

5. Titchmarsh E., "Vvedeniye v teoriyu integralov Fur'ye" (Introduction to the theory of Fourier integrals). - - - Gos. Izd. tekhniko-teoreticho.lit., Moskva - Leningrad, 1948.

6. Buslenko N.P. etal., "Metod statisticheskikh ispytaniy" (the method of statistical analysis). - - - Fizmatgiz, Moskva, 1962.

7. Grenander U. and M. Rosenblatt. Statistical analysis of stationary time series, John Wiley and Sons, 1957.